THOMASON

86

85g - 180 pages

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BY NC SA

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p 2 strongly left enriched

p 3 EQUIMORPHISM = weak equivalence (8 Sept 1995)

p 4 if $K \in \mathcal{K}_{cof}$ $L \rightarrowtail L'$ $\Rightarrow$ $K \otimes L \rightarrowtail K \otimes L'$
strong left enrichment restricts to a left enrichment

p 7 dense $\mathcal{C}_{cof} \subset \mathcal{C}$ need not preserve finite colimits!
eg. free fg $\mathbb{Z}$-mod $= \mathcal{P}$ has all finite colimits,
but $\mathcal{P} \subset \mathbb{Z}$-mod doesn't preserve them
So $\mathcal{C}^s_{cof}(\mathbb{Z}) \subset \mathcal{C}(\mathbb{Z})$ doesn't preserve all finite colimits

p 9: But $\mathcal{C}_{cof} \subset \mathcal{C}$ does preserve cobase change along cofibrations

p 13 Rich hypotheses: enriched over Δ^{op}-sets

p 15 Rich hypotheses self-dual

p 16 Prop: Rich Hypo $\Rightarrow$ $\otimes$ respects colimits,
$\triangleleft$ (colimit) $\times$ limit $\longrightarrow$ limit.

p 24 Rich Hypo $\Rightarrow$ $C \rightarrowtail C'$, $D' \twoheadrightarrow D$ $\quad \underline{Hom}(C', D') \longrightarrow \underline{Hom}(C, D') \times_{\underline{Hom}(C, D)} \underline{Hom}(C', D)$
and $\xrightarrow{\sim}$ if $C \xrightarrow{\sim} C'$ or $D' \xrightarrow{\sim} D$

p 28 Rich Hypo + $\mathcal{N}$ (product) + ω summable
$\Rightarrow$ weak hyp + strong hypo + extra hypo

p 38 $\underline{Hom}(C, \ )$ preserves limits, $\underline{Hom}(\ , C)$ takes colimits to limits

p 40 If $\mathcal{C}$ $\mathcal{N}$-complete, strongly $\mathcal{N}$-reproductive, ultra ω-cosummable
hocolim$_{\Delta/K}$ F exists for each $F: \Delta/K \longrightarrow \mathcal{C}$ (not in $\mathcal{C}_{cof}$ necessarily)

p 44 def of ultra ω-cosummable

p 49 40 holds for Δ^{op}-sets.

p 51 left model, left functorial fact, + enriched over Δ^{op}-sets $\Rightarrow$ $\otimes, D \Rightarrow$
weak, strong + extra hypotheses of §1, §3, §5

p 54 very generally $\underline{Hom}(\text{hocolim}_{\Delta/K} F, C) = \text{holim}_{(\Delta/K)^{op}} \underline{Hom}(F, C)$

$\int^{K} \underline{Hom}(\text{hocolim}_{K/K} FK, GK') \simeq \text{hoend}(K, K' \rightarrow \underline{Hom}_{\mathcal{C}}(FK, GK'))$

hocolim$_{\Delta/K}$ preserves colimits which exist in $\mathcal{C}$ & stay in $\mathcal{C}_{cof}$.
model

p 61. Enriched lift of $\mathcal{N}$ structure from $\mathcal{C}$ to $Cat(K, \mathcal{C})$!
with weak (all needed hypotheses on $\mathcal{C}$ lift to $Cat(K, \mathcal{C})$!)

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so

$$\begin{tikzcd}
 & IIC \arrow[r, "\mu"] & IC \\
\operatorname{hocolim}_{\underline{k}/?}\left(\operatorname{hocolim}_{\underline{k}/k} I'I'C \sqcup \operatorname{hocolim}_{\underline{k}/c} I'(C\cup C)\right) \arrow[r, "\operatorname{holim}(h \to I\circ)"] & \operatorname{hocolim}_{\underline{k}/?}\left(I'I'C \sqcup I'C\cup C\right) \arrow[u, "I(i_0\cup i_1)"] & \\
\operatorname{hocolim}_{\underline{k}/?} I'(C\cup C) \arrow[u, "\text{"}I(i_0\cup i_1)\text{"}"] \arrow[r] & \operatorname{hocolim}_{\underline{k}/?} I'C\cup C \arrow[u, "I i_0 \cup i_1"] &
\end{tikzcd}$$

might possibly work. But will be a mess.

Another ideal: If $\mathcal{C}$ left model category with functorial factorization, and $\mathcal{C}$ left enriched over $\mathcal{A}_f$, then $\exists$ some $\mathcal{A}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$.

Set $K \tilde{\otimes} C$ to $\emptyset \rightarrowtail \tilde{C} \xrightarrow{\sim} C$ functorial.

$K \tilde{\otimes} C =$ ~~pushout~~

$K \otimes \tilde{C}$

cf 80 p 3
reason for weaker left invariant
Now irrelevant

1 H. ~~$K\otimes\tilde{C} \longrightarrow \Delta[0]\otimes\tilde{C} \xrightarrow{\sim} C$~~

$\Delta[0]\tilde{\otimes} C = \Delta[0]\otimes\tilde{C} \xrightarrow{\sim} \tilde{C} \to C$.

Will work if the functorial factor $(\emptyset \to C) \longmapsto \tilde{C}$ is st $C \rightarrowtail C' \Rightarrow \tilde{C} \rightarrowtail \tilde{C}'$ and $\sim$ preserves pushouts along cofibration.

ie want axioms C1, C2, C3 ... cf 79 p 72

I was reluctant to assume

$\mathcal{A}_f \otimes \mathcal{C} \to \mathcal{C}$ for a ull $\mathcal{C}$

because Quillen says (Homotopical Algebra II p 0.2) that certain categories of DGA's don't seem to have a simplicial model structure

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I've glanced through Rational Homotopy and "On the (Co-)homology of commutative rings" without seeing the problem discussed.

Hom enriches show up in Rational Homotopy II 54, seemingly with any previous reference in the page. But is in category of simplicial objects sG.

—

Second reason for being reluctant to assume $1_f \otimes \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ was idea to try to build ~~a struct~~ such a structure from ~~choice of~~ functorial factorization, assuming some $\exists$ ~~$\alpha: I^2 \to I$~~ .. extra axioms. Eg. on ~~Bi~~ 79, 120, 139-141, 159.

Cf. discussion of Problem on 80 p 3

But the approach to constructing an enrichment is now dormant.

~~Imag~~ Any way, if internal preserves cofibration and pushouts, etc, we can use idea of p 1 to extend $1_f \times \mathcal{C}_{cof} \xrightarrow{\otimes} \mathcal{C}_{cof}$ so constructed (if possible) to $1_f \times \mathcal{C} \longrightarrow \mathcal{C}$.

—

So Lets go for

Def.

~~Strongly left enriched over~~

Def. Strongly left enriched (cf p 6 (cf 81 p 51),

Let $\mathcal{K}, \mathcal{L}$ be two left semi-model categories (81 p 48). A strong left enrichment of $\mathcal{L}$ over $\mathcal{K}$ is a functor $\mathcal{K} \times \mathcal{L} \xrightarrow{\otimes} \mathcal{L}$ such that

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LS
~~SK~~ I a 1) $\forall K \in \mathscr{K}$ $K \otimes : \mathcal{Z} \longrightarrow \mathcal{Z}$ sends $\emptyset$ to $\emptyset$ and sends a pushout square along a cofibration in $\mathcal{Z}$ to a pushout square in $\mathcal{Z}$. (I do not ask $K \otimes$ preserves cofibrations in general)

LS I a 2) $\forall L \in \mathcal{Z}$, $\otimes L : \mathscr{K} \longrightarrow \mathcal{Z}$ sends $\emptyset$ to $\emptyset$, and sends pushout squares ~~from~~ along a cofibration to pushout squares in $\mathcal{Z}$. (Do not ask $\otimes L$ preserves cofibrations in general)

LS I 5) EMPTY.

LS I c) If $L \rightarrowtail L'$ is a cofibration in $\mathcal{Z}$ and $K \rightarrowtail K'$ is a cofibration in $\mathscr{K}$, then the pushout

$$K' \otimes L \underset{K \otimes L}{\cup} K \otimes L'$$

is representable in $\mathcal{Z}$, and the induced map

$$K' \otimes L \underset{K \otimes L}{\cup} K \otimes L' \longrightarrow K' \otimes L'$$

is a cofibration.

Moreover, if either $L \rightarrowtail L'$ or $K \rightarrowtail K'$ is also a weak equivalence, then the induced map

$$K' \otimes L \underset{K \otimes L}{\cup} K \otimes L' \longrightarrow K' \otimes L'$$

is a weak equivalence.

Note: Examples show that one should not ask that ~~$K \otimes$~~ $\otimes L$ preserve cofibrations if L not cofibrant. (cf §0 n° 3).

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Dually, we have strongly right enriched.

Lemma Let $\mathcal{K}, \mathcal{L}$ be two left semi-model categories, (SI, 48) and suppose $\mathcal{K} \times \mathcal{L} \xrightarrow{\otimes} \mathcal{L}$ is a strong left enrichment. p. 3.

Then:

1) If $K \in \mathcal{K}_{cof} \subseteq \mathcal{K}$, then

$K \otimes : \mathcal{L} \longrightarrow \mathcal{L}$ preserves cofibrations.

and equicofibrations = cofibrations which are weak equivalences.

2) Moreover, $K \otimes :$ preserves equivalences

(= weak equivalen

New terminology :

"weak equivalences" = equimorphisms [mixes Greek + Latin roots, but no worse than "polyvalent"]

equicofibration = cofibration + weak equivalence

equifibration = fibration and weak equivalen

Lemma Let $\mathcal{K}, \mathcal{L}$ be two left semi-model categories (SI, 48) and $\otimes : \mathcal{K} \times \mathcal{L} \longrightarrow \mathcal{L}$ a strong left enrichment. (p. 3)

Then 1) If $K \in \mathcal{K}_{cof} \subseteq \mathcal{K}$, then $K \otimes : \mathcal{L} \longrightarrow \mathcal{L}$ preserves cofibrations, equicofibrations, and those equimorphisms between cofibrant objects.

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2) If $L \in \mathcal{Z}_{cof} \subseteq \mathcal{Z}$, then $\otimes L : \mathcal{K} \to \mathcal{Z}$

preserves cofibrations, equicofibrations, and those equimorphisms between cofibrant objects.

3) The restriction of $\mathcal{K} \times \mathcal{Z} \xrightarrow{\otimes} \mathcal{Z}$ to $\mathcal{K}_{cof} \times \mathcal{Z}_{cof}$ factors thru $\mathcal{K}$ as $\mathcal{K}_{cof} \times \mathcal{Z}_{cof} \xrightarrow{\otimes_!} \mathcal{Z}_{cof} \to \mathcal{Z}$,

and $\otimes_!$ is a left enrichment in the sense of 81, 51

—

Pf:

1) As $K \in \mathcal{K}_{cof}$, $\emptyset \rightarrowtail K$ is a cofibration.

If $L \rightarrowtail L'$ is a cofibration in $\mathcal{Z}$ (resp. an equicofibration in $\mathcal{Z}$).

$$\begin{array}{ccc} K \otimes L \amalg_{\emptyset \otimes L} \emptyset \otimes L' & \longrightarrow & K \otimes L' \\ \downarrow & & \\ K \otimes L & & \end{array}$$

is a cofibration (resp. equicofibration) by axiom LSIc).

It follows from 85, 115 that $K \otimes$ preserves equimorphisms between cofibrant objects, (Brown's trick cf 80, 11).

2) similar to 1).

3) $\otimes_!$ satisfies the axiom LIa1) of 81, 51

by LSa1) and 1) $\otimes_!$ satisfies the axiom LIa2)

by LSa2) and 2)

and LSIc) implies LIc). QED.

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Lemma. Let $\mathcal{K}$, $\mathcal{L}$ be two left semi-model categories (SI, 7.8) and suppose that every object of $\mathcal{K}$ is cofibrant. Then a functor $\mathcal{K} \otimes \mathcal{L} \xrightarrow{\otimes} \mathcal{L}$ is a strong left enrichment (7.4) iff it satisfies

~~LIa1): (cf SI, 5.1) $\forall K \in \mathcal{K}$~~

sLIa1) (cf SI, 5.1)

$\forall K \in \mathcal{K} = \mathcal{K}_{cof}$, $K \otimes : \mathcal{L} \longrightarrow \mathcal{L}$ preserves initial objects, cofibrations and pushouts along a cofibration.

sLIa2) $\forall L \in \mathcal{L}$, $\otimes L : \mathcal{K} \longrightarrow \mathcal{L}$ sends $\emptyset$ to $\emptyset$, and sends cobase change along a cofibration in $\mathcal{K}$ to a cobase change in $\mathcal{L}$ (but need not preserve the cofibration).

sLIc) If $K \rightarrowtail K'$ is a cofibration in $\mathcal{K}$ and $L \rightarrowtail L'$ is a cofibration in $\mathcal{L}$, then the induced map

$$K' \otimes L \underset{K \otimes L}{\cup} K \otimes L' \longrightarrow K' \otimes L'$$

is a cofibration. It is an equicofibration if in either of the cofibrations $L \rightarrowtail L'$ or $K \rightarrowtail K'$ is also an equimorphism.

(Note the pushout is representable by axiom LB4 of SI, 7.8 as $K \otimes L \rightarrowtail K \otimes L'$ by sLIa1).)

Pf: If $\otimes$ is a strong left enrichment, the sLIa1) holds by SLIa1) and Lemma 1) 7.4. SLIa2) gives SLIa2), and SLIc) gives sLIc).

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Conversely, if SLIa1), SLIa2) or ~~SLIa1~~ SLIc) hold,

Then in the situation of $K \rightarrowtail K'$, $L \rightarrowtail L'$, SLIa1) and left model axiom LBY shows $K' \otimes L \amalg_{K \otimes L} K \otimes L'$ exists a $K \otimes L \rightarrowtail K \otimes L'$. Then SLIc) gives the rest of SLIc). SLIa2) = SLIa2), and SLIa1) is stronger than SLIa1), QED.

Danger. Watch out for colimits in $\mathcal{C}$ of us in $\mathcal{C}$.

Let R be a PID, and $\mathcal{P}$ the category of finitely generated free R-modules.

Then $\mathcal{P}$ has all finite colimits! For clearly, it has finite coproducts. It suffices to show it has coequalizers. As $\mathcal{P}$ is additive, the coequalizer of $P \underset{g}{\overset{f}{\rightrightarrows}} Q$ is the coequalizer of $P \underset{0}{\overset{f-g}{\rightrightarrows}} Q$. So it suffices to show $\mathcal{P}$ has cokernels.

Given $P \xrightarrow{f} Q$ in $\mathcal{P}$, let Q/fP be the cokernel in R-Mod. ~~A~~ ~~B~~ It is finitely generated over R. By the structure theorem for modules over a PID the quotient of Q/fP by its torsion submodule $T(Q/fP) \leq Q/fP$ is free and finitely generated.

If F is any free module and $P \xrightarrow{f} Q \xrightarrow{g} F$ with $gf = 0$, then we have an induced map $Q/fP \longrightarrow F$.

But as F is torsion free, this map factors uniquely

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$Q/fP \twoheadrightarrow (Q/fP)/T(Q/fP) \longrightarrow F.$

So the free $(Q/fP)/T(Q/fP)$ is in fact the cokernel of $P \longrightarrow Q$ in $\mathcal{P}$!!

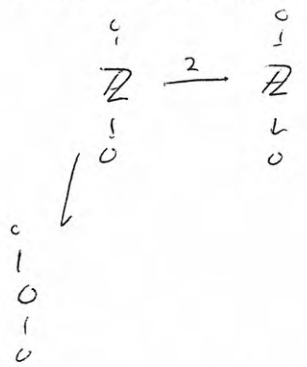
Now let $\mathcal{C}$ be the category of bounded complexes of R-modules of finite type. Give it the left module structure where cofibrations are monomorphisms with degreewise projective cokernel (so degreewise split mono) and where equivalences are homology isomorphisms

Then $\mathcal{C}_{cof} = \mathcal{C}^b(\mathcal{P})$ $\mathcal{P}$ = f.g. projective R-modules.

$\mathcal{C} = \mathcal{C}^b$ (fg. R-modules).

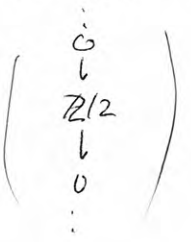
So the above shows $\mathcal{C}_{cof}$ has <u>all</u> <u>finite</u> colimits, (since $\mathcal{P}$ does), but the inclusion $\mathcal{C}_{cof} \longrightarrow \mathcal{C}$ does not preserve them.

Ex.



has pushout $= 0$ in $\mathcal{C}_{cof}(\mathbb{Z})$, but pushout in

$\mathcal{C}(\mathbb{Z})$ is



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Best to hope for is $\mathcal{C}_{cof} \longrightarrow \mathcal{C}$

preserves cobase change along a cof. sn.

Works in above example.

If doesn't work in general, I'll have to be careful to always specify pushouts are to be taken in $\mathcal{C}$ in using old axiom for left enrichment (81, 51)

$$\begin{array}{ccc} C & \rightarrowtail & C' \\ \downarrow & & \\ B & & \end{array} \quad \text{in } \mathcal{C}_{cof}.$$

Take PO in $\mathcal{C}$

$$\begin{array}{ccc} C & \rightarrowtail & C' \\ \downarrow & \square & \downarrow \\ B & \longrightarrow & P \end{array}$$

the LBT $\Rightarrow \quad B \rightarrowtail P. \quad$ So $\quad \emptyset \rightarrowtail B \rightarrowtail P \quad P \in \mathcal{C}_{cf}.$

Since $\mathcal{C}_{cof}$ is a full subcat of $\mathcal{C}$, it forms P in also a pushout in $\mathcal{C}_{cof}$, hence the pushout there.

So is OK.

Lemma Let $\mathcal{C}$ be a left semi-model category (81, 48) and $\mathcal{C}_{cof}$ the full subcategory of cof. brant objects.

Then pushouts along a cof. sn are preserved by the inclusion of $\mathcal{C}_{cof} \longrightarrow \mathcal{C}$. (General pushouts need not be preserved. see example, 7-8).

Thus the inclusion $\mathcal{C}_{cof} \longrightarrow \mathcal{C}$ is a left semi-model functor (85, 29).

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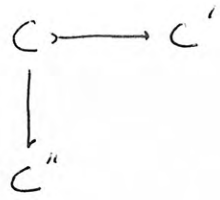
Pf: Let 

be a diagram in $\mathcal{C}_{cof}$. By, the axiom LB4 (§1, 48) the pushout in $\mathcal{C}$ is $C'' \rightarrowtail C' \underset{C}{\sqcup} C''$ a cofibration.

But then $\emptyset \rightarrowtail C'' \rightarrowtail C' \underset{C}{\sqcup} C''$ gives $C' \underset{C}{\sqcup} C'' \in \mathcal{C}_{cof}$.

Since the pushout in $\mathcal{C}$ in fact lies in $\mathcal{C}_{cof}$, and $\mathcal{C}_{cof}$ is a ~~part~~ full subcategory of $\mathcal{C}$, it will also be a pushout in $\mathcal{C}_{cof}$. Since pushouts are unique up to iso, this shows $\mathcal{C}_{cof} \rightarrow \mathcal{C}$ preserves pushouts along a cofibration.

The statement that $\mathcal{C}_{cof} \rightarrow \mathcal{C}$ is a left semi-model functor now follows from the definition (§5, 29) and the method of inducing a left semi model structure on $\mathcal{C}_{cof}$ (§1, 50)

QED.

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Hypoth: Let $\mathcal{C}$ be a left semi-model category, (§1, p 48)

$\mathcal{I}_f$ the category of finite simplicial sets,

and $\mathcal{I}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ a strong left enrichment. (p 2).

Note as every object of $\mathcal{I}_f$ is cofibrant, we may use the equivalent formulation of the conditions on $\otimes$ given by, Lemme p. 6.

—

Def: For $\mathcal{C}$ a left semi-model category, (§1, p 48) strongly left enriched over $\mathcal{I}_f$ (p 2), define

cf §5 p 110.

$$\mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \Delta^{op}\text{-sets}$$

by $\mathrm{Hom}(C, C')_p = \mathrm{Mor}_{\mathcal{C}}(\Delta[p] \otimes C, C')$

—

Ask if $\mathrm{Mor}_{\Delta^{op}\text{Sets}}(K, \mathrm{Hom}(C, C')) \cong \mathrm{Mor}_{\mathcal{C}}(K \otimes C, C')$

—

ok : $K = \Delta\Sigma[1]$.

in general need $K \cong \varinjlim_{n\text{d}\,\Delta/K} \Delta \quad \Rightarrow \quad K \otimes C \cong \varinjlim_{n\text{d}\,\Delta/K} (\Delta[?] \otimes C)$.

ie we

$\otimes C : \mathcal{I}_f \longrightarrow \mathcal{C}$ preserves finite colimits.

—

If C cofibrant ok by §2 p 5 (cf §5 p 110)
§2, 6B

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It looks like this argument can in fact be generalized to work with C not cofibrant.

Key point is for a general C, given

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "po"] & L \arrow[d] \\ N \arrow[r, tail] & P \end{tikzcd}$$

" 1_f

$$\begin{tikzcd} K\otimes C \arrow[r] \arrow[d] & L\otimes C \arrow[d] \\ N\otimes C \arrow[r] & P\otimes C \end{tikzcd}$$

is a pushout square in $\mathcal{C}$, ~~...~~ by axiom LSIa1),

even though $K\otimes C \to L\otimes C$ need not be a cofibration if C not cofibrant.

In ~~...~~ particular $\forall N$ ~~$\coprod_N \Delta \ldots = (\coprod_N \Delta[0]) \otimes C$~~

$\coprod_N (\Delta[0]\otimes C)$ exists in C as a ~~sum~~ coproduct

$\left(= (\coprod_N \Delta[0]) \otimes C \right)$.

—

But this is getting complicated.

By the time we add the extra hypotheses of p ~~137~~ 136 we'll wind up with an enrichment over Δ^{op} sets ~~...~~ ~~...~~ anyway.

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So just go for it. Will be simpler.

~~Natural Hypotheses:~~

~~$(\mathcal{C}, \text{cof.s}\,\mathcal{C}, \text{equi}\,\mathcal{C}, \text{f.s}\,\mathcal{C})$ is a basic model category with functorial factorization~~

~~Enrich~~

~~Natural Hypoths~~ Rich Hypotheses

$(\mathcal{C}, \text{cof.s}\,\mathcal{C}, \text{equi}\,\mathcal{C}, \text{f.s}\,\mathcal{C})$ a basic model category

i.e. satisfies axioms B1 – B7 of 78 p156-157.

Suppose $\mathcal{C}$ is enriched over Δ^{op}-sets in the usual sense of category theory:

$$\underline{\text{Hom}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \underline{\Delta^{op}\text{-sets}} \quad \text{s.t. } \ldots$$

Let $\mathcal{A}_f \subseteq \underline{\Delta^{op}\text{-sets}}$ be the full subcategory of those simplicial sets with only finitely many non-degenerate simplices (i.e. "finite simplicial sets")

We suppose further that there are functors

$$\mathcal{A}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} \qquad \mathcal{A}_f^{op} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$$

Better notation p19

~~Map(,)~~ ~~$\mathcal{A}_f^{op} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$~~

such that there are natural isomorphisms for $K \in \mathcal{A}_f$, $C, C' \in \mathcal{C}$

$$\underline{\text{Hom}}(K \otimes C, C') \cong \text{Map}(K, \underline{\text{Hom}}(C, C')) \cong \underline{\text{Hom}}(C, K \triangleright C')$$

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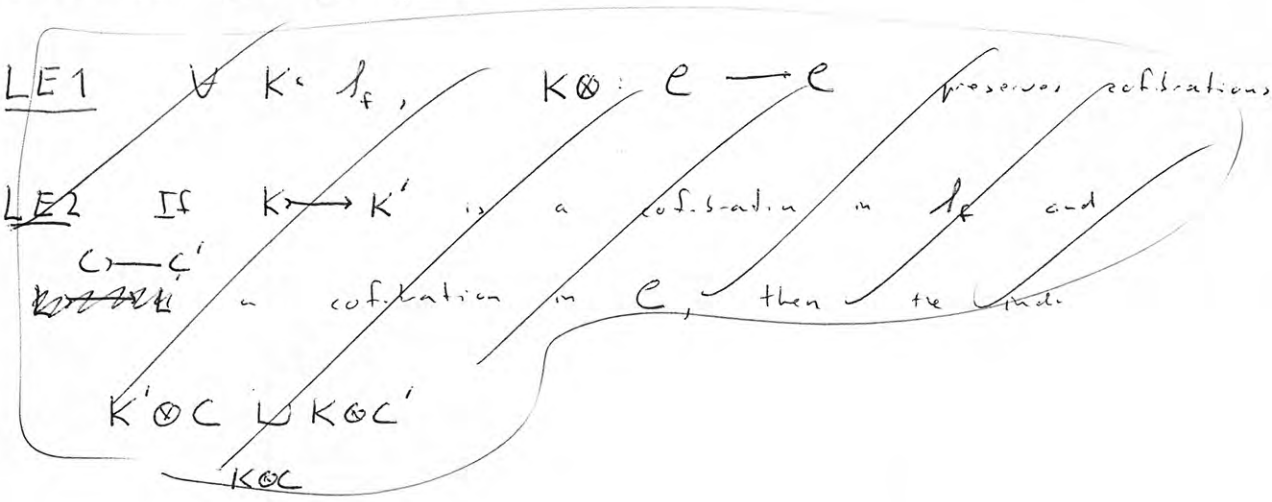
Moreover, we suppose

~~LE~~ ~~If $K \rightarrowtail K'$ is a cofibration in $\mathcal{I}_f$ and $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$~~



LE1 $\forall K$, $\mathcal{I}_f$, $K \otimes : \mathcal{C} \longrightarrow \mathcal{C}$ preserves cofibrations.

LE2 If $K \rightarrowtail K'$ is a cofibration in $\mathcal{I}_f$ and $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$, then the induced map $K' \otimes C \underset{K \otimes C}{\sqcup} K \otimes C' \longrightarrow K' \otimes C'$ is a cofibration in $\mathcal{C}$.

LE If $K \rightarrowtail K'$ is a cofibration in $\mathcal{I}_f$ and $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$, then the pushout $K' \otimes C \underset{K \otimes C}{\sqcup} K \otimes C'$ is representable in $\mathcal{C}$, and the induced map $K' \otimes C \underset{K \otimes C}{\sqcup} K \otimes C' \longrightarrow K' \otimes C'$ is a cofibration in $\mathcal{C}$.

It is a equivalence if either of the cofibrations is an equivalence.

Remark: The hypothesis that $K' \otimes C \underset{K \otimes C}{\sqcup} K \otimes C'$ is representable is in fact redundant. For $\emptyset \otimes C \cong \emptyset$ (follows from adjunct. $\underline{\mathrm{Hom}}\, \mathrm{Mor}_{\mathcal{C}}(\emptyset \otimes C, C') = \underline{\mathrm{Hom}}(\emptyset \otimes C, C')_0 = \mathrm{Map}(\emptyset, \underline{\mathrm{Hom}}(C, C'))_0 = *$.)

and $\forall K'$ $K' \otimes C \underset{\emptyset}{\sqcup} \emptyset = K' \otimes C$. ~~Thus we set LE =>~~

Since $\forall K'$, $\emptyset \rightarrowtail K' \in \mathcal{I}_f$, LE gives this instance of LE gives $K' \otimes$ preserves cofibrations.

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But then in the general case $C \rightarrowtail C' \Rightarrow K'\otimes C \rightarrowtail K'\otimes C'$,

and now axiom B4 for basic model cats (or LB4 for left model cat)

gives the pushout exist in general.)

RE If $K \rightarrowtail K'$ is a cofibration in Δ_f and $C \twoheadrightarrow C'$ is a fibration in $\mathcal{C}$, then

$$K' \overset{\Delta}{\pitchfork} C' \underset{K \overset{\Delta}{\pitchfork} C'}{\times} K \overset{\Delta}{\pitchfork} C \quad \text{is represented in } \mathcal{C}$$

and the induced map

$$K' \overset{\Delta}{\pitchfork} C \longrightarrow K' \overset{\Delta}{\pitchfork} C' \underset{K \overset{\Delta}{\pitchfork} C'}{\times} K \overset{\Delta}{\pitchfork} C \quad \text{is a fibration.}$$

This map is an equifibration if either the cofibration $K \rightarrowtail K'$ or the fibration $C \twoheadrightarrow C'$ is an equimorphism.

[the dual to remark LE applies to RE.]

Lemma: If $(\mathcal{C}, \mathrm{cof}\,\mathcal{C}, \mathrm{equi}\,\mathcal{C}, \mathrm{fib}\,\mathcal{C}, \underline{\mathrm{Hom}}, \otimes, \overset{\Delta}{\pitchfork})$ satisfies the hypotheses of p 13, then the dual model category $(\mathcal{C}^{op}, \mathrm{cof}\,\mathcal{C}^{op} = (\mathrm{fib}\,\mathcal{C})^{op}, \mathrm{equi}\,\mathcal{C}^{op} = (\mathrm{equi}\,\mathcal{C})^{op}, \mathrm{fib}\,\mathcal{C}^{op} = (\mathrm{cof}\,\mathcal{C})^{op})$ (cf 78, p 159)

with $\underline{\mathrm{Hom}}^{op} : (\mathcal{C}^{op})^{op} \times \mathcal{C}^{op} = \mathcal{C} \times \mathcal{C}^{op} \cong \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\underline{\mathrm{Hom}}} \Delta^{op}$-sets

(i.e. $\underline{\mathrm{Hom}}^{op}(C', C) = \underline{\mathrm{Hom}}(C, C')$, etc.).

[cf Eilenberg–Kelley III 2.1]

~~$K\otimes\mathcal{C}^{op}$~~ $\Delta_f \times \mathcal{C}^{op} \xrightarrow{\otimes^{op}} \mathcal{C}^{op}$

$$= (\Delta_f^{op} \times \mathcal{C})^{op} \xrightarrow{\overset{\Delta}{\pitchfork}} \mathcal{C}^{op},$$

$$\overset{\Delta}{\pitchfork}{}^{op} : \Delta_f^{op} \times \mathcal{C}^{op} \longrightarrow \mathcal{C}^{op}$$
$$\| \qquad\qquad \|$$
$$(\Delta_f \times \mathcal{C})^{op} \xrightarrow{\otimes} \mathcal{C}^{op}$$

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satisfies the the rich hypotheses of p13.

Thus the rich hypotheses are invariant under duality.

—

Prop Suppose the rich hypotheses of p.13, except LE or RE.

Then

1) There is a natural identification preserving composition and units

$$\cancel{[\underline{\mathrm{Hom}}(C,C')]_0 = \mathrm{Map}(\Delta[0]\ [\mathrm{Map}}$$

$$[\underline{\mathrm{Hom}}(C,C')]_0 \cong \mathrm{Mor}_e(C,C')$$

~~2) There are compatible natural isomorphisms compo~~

2) There are natural isomorphisms

$$\Delta[0]\otimes C \cong C \qquad \Delta[0]\pitchfork C' \cong C'$$

render commutative the diagram

$$\begin{array}{ccccc}
[\underline{\mathrm{Hom}}(\Delta[0]\otimes C, C')]_0 & \cong & [\mathrm{Map}(\Delta[0], \underline{\mathrm{Hom}}(C,C')]_0 & \cong & [\underline{\mathrm{Hom}}(C, \Delta[0]\pitchfork C')]_0 \\
\wr\| & & \wr\| & & \wr\| \\
\mathrm{Mor}_e(\Delta[0]\otimes C, C') & & [\underline{\mathrm{Hom}}(C,C')]_0 & & \mathrm{Mor}_e(C, \Delta[0]\pitchfork C') \\
& \cong & \wr\| & \cong & \\
& & \mathrm{Mor}_e(C,C') & &
\end{array}$$

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3) If $K : \underline{\underline{D}} \longrightarrow \mathcal{A}_f$ is any diagram such that the

~~$\varinjlim_{\underline{\underline{D}}} K$ exists in $\mathcal{A}_f$ taken in $\mathcal{A}$~~ colimit taken in ~~$\mathcal{A}$~~ Δ^{or}-sets

$\varinjlim_{\underline{\underline{D}}} K$ lies in fact in $\mathcal{A}_f$, then for all $C \in \mathcal{C}$ ~~the natural~~

~~map then~~ $(\varinjlim_{\underline{\underline{D}}} K) \otimes C$ is the colimit in $\mathcal{C}$ of

the diagram $K \otimes C : \underline{\underline{D}} \longrightarrow \mathcal{C}$.

(Maybe $\mathcal{A}_f$ — Δ^{or}-sets preserves all colimits that exist in $\mathcal{A}_f$ ~~$\mathcal{A}$~~. ??).

Yes but proof ugly. Not worth it.

Dually if $L : \underline{\underline{D}}^{op} \longrightarrow \mathcal{A}_f$ is any diagram such that

the colimit in Δ^{or}set $\varinjlim_{\underline{\underline{D}}^{op}} L$ lives in fact in $\mathcal{A}_f$, then $\forall C' \in \mathcal{C}$

~~$\varinjlim_{\underline{\underline{D}}^{op}} L$~~ $(\varinjlim_{\underline{\underline{D}}^{op}} L) \overset{\triangleright}{\cancel{\otimes}} C'$ represents the limit of

~~$\underline{\underline{D}} \longrightarrow \mathcal{C}$~~ $L \overset{\triangleright}{\cancel{\otimes}} C' : \underline{\underline{D}} \longrightarrow \mathcal{C}$.

4) If ~~$K : \mathcal{A}$~~ $F : \underline{\underline{D}} \longrightarrow \mathcal{C}$ ~~a~~ is any diagram

with a colimit $\varinjlim_{\underline{\underline{D}}} F$ in $\mathcal{C}$, then $\forall K \in \mathcal{A}_f$

~~the natural map~~ $K \otimes (\varinjlim_{\underline{\underline{D}}} F)$ represents the

colimit of $K \otimes F : \underline{\underline{D}} \longrightarrow \mathcal{C}$ $(D \longmapsto K \otimes FD)$.

~~8~~ Dually if $F' : \underline{\underline{D}} \longrightarrow \mathcal{C}$ is any diagram with a

limit $\varprojlim_{\underline{\underline{D}}} F'$ in $\mathcal{C}$, then $\forall K \in \mathcal{A}_f$ $K \overset{\triangleright}{\cancel{\otimes}} (\varprojlim_{\underline{\underline{D}}} F')$

represents the limit of $K \overset{\triangleright}{\cancel{\otimes}} F' : \underline{\underline{D}} \longrightarrow \mathcal{C}$.

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Pf 1) Follows from

$$[\underline{\mathrm{Hom}}(C,C')]_0 = \mathrm{Map}(X,Y) \qquad \mathrm{Mor}_{\Delta^{op}\mathrm{Set}}(X,Y)$$

and standard properties of enrichment over a ~~cat~~ symmetric ~~closed category~~ monoidal closed category. (cf Kelly, Basic ~~Concepts~~ Concepts of Enriched Category Theory 1.3)

2) We have obtained from the adjunction isos natural isos

$$\mathrm{Mor}_{\mathcal{C}}(C \otimes \Delta[0], C') \cong \mathrm{Mor}_{\mathcal{C}}(C, C') \cong \mathrm{Mor}_{\mathcal{C}}(C, \Delta[0] \pitchfork C').$$

That these are induced by unique ~~isos~~ natural isos

$$C \otimes \Delta[0] \cong C, \qquad C' \cong \Delta[0] \pitchfork C'$$

now follows from Yoneda

3) ~~$\underline{\mathrm{Hom}}(K \otimes C, C') \cong \mathrm{Mor}(K, \underline{\mathrm{Hom}}(C, C'))$~~

$$\underline{\mathrm{Hom}}(K \otimes C, C') \cong \mathrm{Map}(K, \underline{\mathrm{Hom}}(C, C'))$$

implies on taking Γ_0 and using 1)

a natural iso

$$\mathrm{Mor}_{\mathcal{C}}(K \otimes C, C') \cong \mathrm{Mor}_{\Delta^{op}\mathrm{Set}}(K, \underline{\mathrm{Hom}}(C, C')).$$

Then $\mathrm{Mor}_{\mathcal{C}}((\varinjlim_{\mathcal{D}} K) \otimes C, C') \cong \mathrm{Mor}_{\Delta^{op}\mathrm{Set}}(\varinjlim_{\mathcal{D}} K, \underline{\mathrm{Hom}}(C, C'))$

$$\varprojlim_{\mathcal{D}^{op}} \mathrm{Mor}_{\mathcal{C}}(K \otimes C, C') \cong \varprojlim_{\mathcal{D}^{op}} \mathrm{Mor}_{\Delta^{op}\mathrm{Set}}(K, \underline{\mathrm{Hom}}(C, C'))$$

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This shows $(\varinjlim_{\mathcal{D}} K) \otimes C$ represents the colimit, by definition.

(Note $K \otimes C$ as $I_f \longrightarrow C$ is not left adjoint to $\underline{Hom}(C, \ )$, as the latter goes to Δ^{op} Sets, not to I_f !)

Dual half of 3) is similar, & or follows by duality!

4) We have

$$Mor_{\mathcal{C}}(K \otimes C, C') \cong Mor_{\Delta^{op}\text{s}}(K, \underline{Hom}(C, C')) \cong Mor_{\mathcal{C}}(C, K \pitchfork C')$$

So $K \otimes$ is left adjoint to $K \pitchfork : \mathcal{C} \longrightarrow \mathcal{C}$, and hence preserves colimits. (or argue as in part 3)).

Dually, $K \pitchfork$ preserves whatever limits that may exist

QED

Notation. $K \pitchfork X$ looks too symmetric, and is clunky.

$Mor(K, X)$ too long-winded, and obscures duality.

Better candidates (exist in TeX

(Symbols which exist in AmsTeX,

$\supset$	\Supset	all binary relations
$\sqsupset$	\sqsupset	(available binary operators too symmetric)
$\triangleright$	\vartriangleright	
$\propto$	\varpropto	
$\blacktriangleright$	\blacktriangleright	all in Further Funb. use \Use AMS symbols

$\propto$	\propto	standard ams fm
$\nsubseteq$		
$\sqsubseteq$	\sqsubseteq	

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$\sqsupset$ \sqsupset available in latexsym package.

$\sqsupseteq$ \sqsupseteq in plain latex.

$\sqsupseteq$ \sqsupseteq in TeX.

~~TeX has $\triangleright$ \triangleright right a~~

TeX (

$\triangleright$ = \triangleright ~~adv~~, even is binary op.

I. ~~texte~~ Latex

Bisse $\triangleright$ = \rhd needs latexsym package

~~Amstex~~.

Amstex seems not to have a bis $\triangleright$ as binary operator

But could use $\triangleright$ \vartriangleright in

<u>Conclusion</u>

Write ~~adj~~ enriched adjunction.

$$\mathrm{Map}(K, \underline{\mathrm{Hom}}(C, C')) \cong \underline{\mathrm{Hom}}(C, K \triangleright C')$$

~~ie~~ $K \triangleright C'$ is the cofensor $(\ell_f)^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$

$A \triangleright B$ like $\mathrm{Map}(A, B) = \{A \to B\}$.

In Latex ~~is~~ use binary operator $\triangleright$ = \rhd
in ~~tex~~ latexsym package.

OK for AMS Latex ?

In ~~TeX~~ AMS TeX, use binary relation $\triangleright$ = \vartriangleright
~~defined~~ retyped to be a binary operator.

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In plain Tex use the dinky binary operat. $\triangleright$ = \triangleright.

Arghh But maybe $\Delta[n] \triangleright X$ will look bad.

If so fall back to $\Delta[n] \pitchfork X$

Lemma: Let $(\mathcal{C}, \text{cofib}\,\mathcal{C}, \text{equiv}\,\mathcal{C}, \text{fib}\,\mathcal{C}, \underline{\text{Hom}}, \otimes, \triangleright)$ satisfy the ~~hypotheses~~ rich hypotheses of §3 except for possibly LE and RE.

~~Then LE and RE~~

Then the ~~conclusions~~ LE and RE are each seperately equivalent to

EB: If $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$ and $D \twoheadrightarrow D'$ is a fibration the induced map

$$\underline{\text{Hom}}(C', D) \longrightarrow \underline{\text{Hom}}(C, D) \times_{\underline{\text{Hom}}(C, D')} \underline{\text{Hom}}(C', D')$$

EB: If $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$ and $D' \twoheadrightarrow D$ is a fibration in $\mathcal{C}$, the induced map

$$\underline{\text{Hom}}(C', D') \longrightarrow \underline{\text{Hom}}(C, D') \times_{\underline{\text{Hom}}(C, D)} \underline{\text{Hom}}(C', D)$$

is a Kan fibration in Δ^{op}sets.

It is a "weak equivalence" if either the cofibration $C \rightarrowtail C'$ or the fibration $D' \twoheadrightarrow D$ is also an equivalence.

Pf: In particular LE holds iff RE holds.

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No Quillen's version (Homotopical Algebra II 2.3 Prop 3)

depends on characterizing fibrations as morphisms with the right lifting property for equicofibrations etc

—

What I have is

cf Quillen Homotopical Algebra

Lemma: Let $(\mathcal{C}, \mathrm{cof}\mathcal{C}, \mathrm{equi}\mathcal{C}, \mathrm{fib}\mathcal{C})$ ~~basic model~~ category satisfy B1 – B6 of the basic model axioms (ie no factorization axiom) § 78, 156

Let $\mathcal{C}$ be enriched over Δ°-sets via $\underline{\mathrm{Hom}}$, and suppose $\mathcal{C}$ has a tensor $\otimes$ and cotensor $\triangleright$ over $I_f \subseteq \Delta^{\circ}$-Sets s.t. there are natural isomorphisms

for $K \subseteq I_f$, $C, C' \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(K \otimes C, C') \cong \mathrm{Map}(K, \underline{\mathrm{Hom}}(C, C')) \cong \underline{\mathrm{Hom}}(C, K \triangleright C)$$

Then the ~~combination~~ conjunction of axioms LE and RE (p. 14) is equivalent to the following:

EB If $C \rightarrowtail C'$ is a cofibration in $\mathcal{C}$ and $D' \twoheadrightarrow D$ is a fibration, then the induced map

$$\underline{\mathrm{Hom}}(C', D') \longrightarrow \underline{\mathrm{Hom}}(C, D') \times_{\underline{\mathrm{Hom}}(C, D)} \underline{\mathrm{Hom}}(C', D)$$

is a Kan fibration of simplicial sets. Moreover if "a "weak equivalence" in the usual sense if either the cofibration $C \rightarrowtail C'$ or the fibration $D' \twoheadrightarrow D$ is an equimorphism in $\mathcal{C}$.

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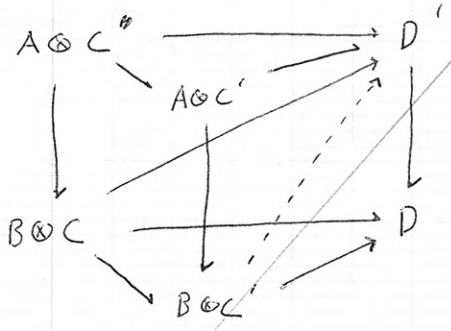
Pf:

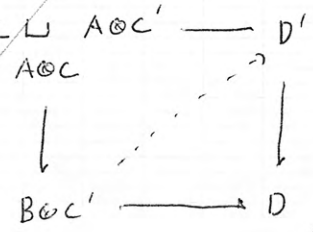
First note $f : A \to B$ in Δ^{op}-sets, the adjunction implies a bijective correspondence between solid diagrams (solid or solid plus dotted arrows)

$$\begin{array}{ccc} A & \longrightarrow & \underline{Hom}(C', D') \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \underline{Hom}(C, D') \times \underline{Hom}(C', D) \\ & & \underline{Hom}(C, D) \end{array}$$

and

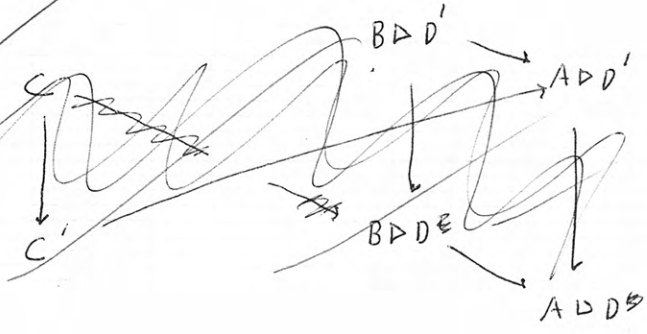


ie



both the adjunction!

Similarly, these correspond injectively to



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cf §5 p.112
Quillen Homotopical Alg. Prop 3
II 2.3

Lemma Let $(\mathcal{C}, \mathrm{cof}\mathcal{C}, \mathrm{equi}\mathcal{C}, \mathrm{fib}\mathcal{C})$ satisfy B1 – B6 of §7 p.156, ie the basic model axioms except factorization.

Suppose $\mathcal{C}$ is enriched over Δ^{op}-sets, with internal Hom. $\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \Delta^{op}$-sets.

Suppose there is a functor

$$\mathcal{A}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

and a natural isomorphism

for $K \in \mathcal{A}_f$, $C, C' \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(K \otimes C, C') \cong \underline{\mathrm{Map}}(K, \underline{\mathrm{Hom}}(C, C'))$$

If $\otimes$ satisfies LE (p.141), then

BE:

for each cofibration $C \rightarrowtail C'$ in $\mathcal{C}$ and each fibration $D' \twoheadrightarrow D$ in $\mathcal{C}$, the induced map

$$\underline{\mathrm{Hom}}(C', D') \longrightarrow \underline{\mathrm{Hom}}(C, D') \times_{\underline{\mathrm{Hom}}(C, D)} \underline{\mathrm{Hom}}(C', D)$$

is a Kan fibration of simplicial sets.

It is a weak equivalence in the usual sense if either the cofibration $C \rightarrowtail C'$ or the fibration $D' \twoheadrightarrow D$ is an equimorphism.

(Dually, if there is a $\mathcal{A}_f^{op} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$ such that $\underline{\mathrm{Map}}(K, \underline{\mathrm{Hom}}(C, C')) \cong \underline{\mathrm{Hom}}(C, K \triangleright C')$ naturally, and $\triangleright$ satisfies RE, then EB holds.

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Pf: The dual statement indeed is dual, ie the given statement for $\mathcal{C}^{op}$.

So it suffices to prove the given statement.

The ~~adjunction~~ isomorphism

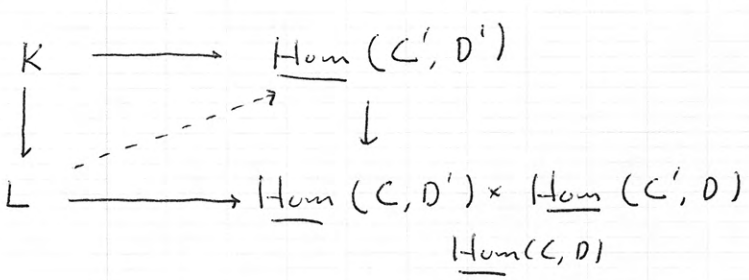
$$\underline{\mathrm{Hom}}(K \otimes C, C') \cong \underline{\mathrm{Hom}}(K, \underline{\mathrm{Hom}}(C, C'))$$

implies on taking $[\ ]_0$ on both

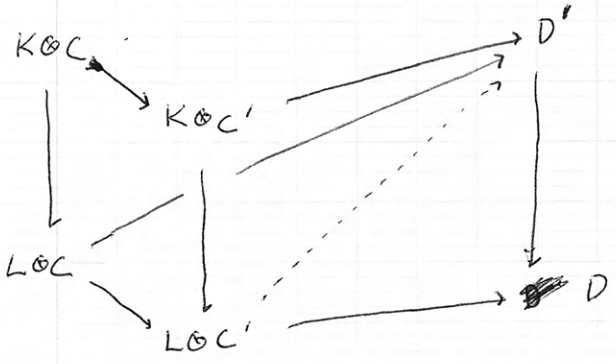
$$\mathrm{Hom}_{\mathcal{C}}(K \otimes C, C') \cong \mathrm{Hom}_{\Delta^{op}\mathrm{Set}}(K, \underline{\mathrm{Hom}}(C, C')).$$

This induces a natural bijection between diagrams of solid arrows (resp. solid and dotted arrows) in ~~$\mathcal{C}^{op}$~~ ~~from $K \to \underline{\mathrm{Hom}}(C, D)$~~

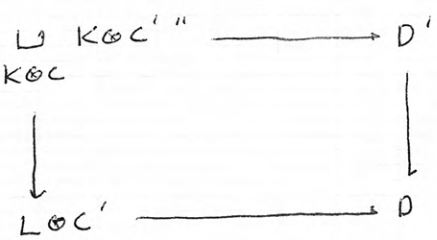
$K \to L$ in Δ^{op}-set.



cf §5, 118



ie



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Now to say $\underline{Hom}(C', D') \longrightarrow \underline{Hom}(C, D') \times_{\underline{Hom}(C,D)} \underline{Hom}(C', D)$

is a Kan fibration is to say for all horn $\Lambda[n] \xrightarrow{\sim} \Delta[n]$

one can always find a ~~sol~~ dotted arrow filling in an

solid arrow diagram of the form

$$\begin{array}{ccc} \Lambda[n] & \longrightarrow & \underline{Hom}(C', D') \\ \sim\downarrow & \nearrow & \downarrow \\ \Delta[n] & \longrightarrow & \underline{Hom}(C, D') \times_{\underline{Hom}(C,D)} \underline{Hom}(C', D) \end{array}$$

But the corresponding diagram is

$$\begin{array}{ccc} \Delta[n] \otimes C \sqcup_{\Lambda[n]\otimes C} \Lambda[n] \otimes C' & \longrightarrow & D' \\ \downarrow & \nearrow & \downarrow \\ \Delta[n] \otimes C' & \longrightarrow & D \end{array}$$

By LE, the left vertical map is a equivalence.

Then by axiom B6, the fill-in arrow exists. Thus

the fill-in exists in the corresponding diagram.

~~If~~ Suppose now either $C \hookrightarrow C'$ or $D' \twoheadrightarrow D$ is an

equivalence. To show the ~~Kan fibration~~ map is Δ^n-sat

$$\underline{Hom}(C', D') \longrightarrow \underline{Hom}(C, D') \times_{\underline{Hom}(C,D)} \underline{Hom}(C', D)$$

is a Kan fibration and weak equivalence

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it suffices to show ~~any solid arrow diagram of $\mathring{\Delta}[n] \to \Delta[n]$~~ for all n on $\mathring{\Delta}[n] \rightarrowtail \Delta[n]$ that any solid arrow diagram of the form

$$\begin{tikzcd}
\mathring{\Delta}[n] \arrow[r] \arrow[d, tail] & \underline{\mathrm{Hom}}(C', D') \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(C, D') \times_{\underline{\mathrm{Hom}}(C, D)} \underline{\mathrm{Hom}}(C', D)
\end{tikzcd}$$

admits a dotted fill-in. (cf Quillen Homotopical Algebra II 2.2 Def 1, II 3.4 Thm 3, II 3.19 Prop 7)

But the corresponding diagram is

$$\begin{tikzcd}
\Delta[n] \otimes C \underset{\mathring{\Delta}[n] \otimes C}{\sqcup} \mathring{\Delta}[n] \otimes C' \arrow[r] \arrow[d] & D' \arrow[d] \\
\Delta[n] \otimes C' \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

~~But~~ LE gives that $\Delta[n] \otimes C \underset{\mathring{\Delta}[n] \otimes C}{\sqcup} \mathring{\Delta}[n] \otimes C' \longrightarrow \Delta[n] \otimes C'$ is a cofibration, and an equicofibration ~~wh~~ if $C \rightarrowtail C'$ is an equimorphism. Thus either the left cofibration or the right fibration $D' \twoheadrightarrow D$ is an equimorphism, and the dotted arrow exists by axiom BM6.

QED.

Remark: Conversely, BE implies $\forall\ K \rightarrowtail L$ "Δ^{op}-sch ~~that~~ $\forall\ C \rightarrowtail C'$ in $\mathcal{C}$, then ~~[illegible]~~ if

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~~K~~ "$L\otimes C \sqcup_{K\otimes C} K\otimes C'$" is representable in $\mathcal{C}$, then

$L\otimes C \sqcup_{K\otimes C} K\otimes C' \longrightarrow L\otimes C'$ has the left

lifting property for trivial fibrations.

But with ~~so~~ my basic model axioms, ~~the~~ unlike Quillen's, this does not imply $L\otimes C \sqcup_{K\otimes C} K\otimes C' \text{---} L\otimes C'$ is a cofibration.

So I can't deduce LE from EB.

cf Quillen, Homotopical Algebra II 2.3 Prop 3)

---

cf. 38

cf 54(a) one-sin

Lemma:

Let $(\mathcal{C}, \text{cofib}\,\mathcal{C}, \text{equi}\,\mathcal{C}, \text{fib}\,\mathcal{C})$ be a basic model category with functorial factorization

(ie satisfies B1–B6 ~~of~~ a F1, F1' of 78 p 156–158: here B7 note this is slight redundant in presence of enrichment below, which should allow construction of cylinders, eg. cf 79 p 72 C1 + enrich ⇒ C2 ⇒ F1, F1' ?)

Suppose $\mathcal{C}$ is enriched over Δ^{on}-sets by Hom, and $\exists$ $\Lambda_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ and $\Lambda_f^{\text{on}} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$

with natural isomorphisms

$$\underline{\text{Hom}}(K\otimes C, C') \cong \underline{\text{Map}}(K, \underline{\text{Hom}}(C, C')) \cong \underline{\text{Hom}}(C, \cancel{C'}\, K \triangleright C')$$

~~ie st the~~

and st RE and LE of p 14–15 hold

(ie $\mathcal{C}$ satisfies the Rich Hypothesis of p 13)

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~~Suppose that $\mathcal{C}$ is~~

Let λ be an infinite cardinal, and suppose that $\mathcal{C}$ is λ-productive (resp. strongly λ-productive) and ω-summable (resp. strongly ω-summable).

(SI, p. 118, p. 117).

Then

The underlying right model category of $\mathcal{C}$ w.r.t. $\mathbb{1}_f^{\omega} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$

satisfies

a) There is a natural isomorphism $\forall\ C$

$$C \cong \Delta[\omega] \triangleleft C$$

b) For all indexing sets I with $\#I < \lambda$, and all families $\{C_i\}_{i \in I}$ in $\mathcal{C}_{f.s.}$ (resp. $\mathcal{C}$)

and all $K \in \mathbb{1}_f$ the natural map

$$K \triangleright \left(\prod_I C_i\right) \longrightarrow \prod_I (K \triangleright C_i)$$

is an isomorphism.

c) $\forall\ C, C' \in \mathcal{C}$, $\forall\ K, L \in \mathbb{1}_f$ there are natural isomorphisms

$$K \triangleright (L \triangleright C)$$

SII

$$((K \times L) \triangleright C)$$

SII

$$(L \times K) \triangleright C$$

SII

$$L \triangleright (K \triangleright C)$$

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c) $\forall\, C \in \mathcal{C}$, $\forall K, L \in \mathcal{I}_c$ are in a natural isomorphism

$$K \triangleright (L \triangleright C) \cong (K \times L) \triangleright C.$$

This isomorphism and the symmetry of $\times$ in $\mathcal{I}_c$ induces

$$\gamma : K \triangleright (L \triangleright C) \cong L \triangleright (K \triangleright C)$$

satisfying the requirements of §5, 141 NEX, §5, 150 c).

The iso $K \triangleright (L \triangleright C) \cong (K \times L) \triangleright C$ satisfies the commutativity condition of §5, 149 (dual to extra hypotheses §5, 136).

d) If $\cdots \to C_n \to \cdots \to C_0$ is a tower of fibrations in $\mathcal{C}_{f.s}$ (resp. in $\mathcal{C}$), then $\forall K \in \mathcal{I}_c$ the natural map

$$K \triangleleft (\varprojlim_{\omega} C_i) \longrightarrow \varprojlim_{\omega} (K \triangleleft C_i)$$

is an isomorphism

e) Then $\mathcal{C}$ with $\triangleleft : \mathcal{I}_c^{op} \times \mathcal{C}_{f.s} \to \mathcal{C}_{f.s}$

satisfies: the duals of

the weak hypotheses of §1, 143

the strong hypotheses of §3, 143

and the extra hypotheses of §5, 136

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Pf: a) " Prop 2) p16.

b) ~~If all C_i are~~

For $C_i \in \mathcal{C}_0$ we have $\emptyset \to K$, $C_i \to *$

By RE this gives $K \triangleright C_i \to \emptyset \triangleright C_i \times_{\emptyset \triangleright *} K \triangleright * \simeq * \times_{*} * = *$

is a fibration, so $K \triangleright C_i$ is fibrant. Thus ~~if~~ in the case of $\mathcal{K}$-products, all the C_i, hence all $K \triangleright C_i$ are fibrant, so

~~If~~ $\prod_I K \triangleright C_i$ exists. Then the case of strong $\mathcal{K}$-products

for an $\{C_i\}_{i \in I}$ $\#I \in \mathcal{K}$, $\prod_I K \triangleright C_i$ exists.

not really need

or the implies exists.

Now Prop 4) p16 gives $K \triangleright (\prod_I C_i)$ represented $\prod_I (K \triangleright C_i)$

So the canonical map is an iso.

c) For all $C' \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(C', K \triangleright (L \triangleright C)) \cong \underline{\mathrm{Map}}(K, \underline{\mathrm{Hom}}(C', L \triangleright C))$$

$$\cong \underline{\mathrm{Map}}(K, \underline{\mathrm{Map}}(L, \underline{\mathrm{Hom}}(C', C))$$

$$\cong \underline{\mathrm{Map}}(K \times L, \underline{\mathrm{Hom}}(C', C))$$

$$\cong \underline{\mathrm{Hom}}(C', (K \times L) \triangleright C)$$

Taken $\lceil \rceil_0$ we get $K \triangleright (L \triangleright C)$ or $(K \times L) \triangleright C$ representing the same functor, so they are isomorphic by ~~Yoneda~~ Yoneda.

The compatibilities required by $\underline{\S 5}$ p41 NEX, $\underline{\S 5}$ p50 c) and $\underline{\S 5}$ p49 follow from Prop 2) p16 and computations in the ~~closed~~ symmetric monoidal closed model category $\underline{\Delta^{op}\text{-sets}}$

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d) Follows from Prop 4) p16.

e) As to the dual weak Hypotheses 81 p143

1) follows by forgetting part the model structure ~~(cf 81 p143~~

(cf the statement of 1) 81 p143 !)

2) ~~Then enrich~~

If $K \in \mathcal{I}_f$ $C \in \mathcal{C}_{cof}$ in $\emptyset \rightarrowtail K$ ($\mathcal{I}_{f\,cof} = \mathcal{I}_f$)

and $\emptyset \rightarrowtail C$ in LE gives $\emptyset \rightarrowtail K \otimes C \in$ ~~$\mathcal{C}_{cof}$~~.

so $K \otimes C \in \mathcal{C}_{cof}$ (not $\emptyset \otimes C = \emptyset = K \otimes \emptyset$ by Prop 3), 4) p16

~~Then the~~

so $\mathcal{I}_f \times \mathcal{C}_{cof} \xrightarrow{\otimes} \mathcal{C}_{cof}$.

Now the left enriched axioms follow (cf 81 p51

LIa1) of ~~81~~ by Prop 3) p16 + LE gives $\forall K \in \mathcal{I}_f$ $\emptyset \rightarrowtail K$

LIa2) follows by Prop 4) p16 + LE gives $\forall C \in \mathcal{C}_{cof}$, $\emptyset \rightarrowtail C$.

LIc) is just LE.

And $\mathcal{C}$ is λ-

e) As to the duals of

the weak hypotheses of 81 p143 (cf 85 p145)

1) holds by just forgetting the left part of the model structure (cf the very statement of 81 p143 #1).

2) If $K \in \mathcal{I}_f$ and $C \in \mathcal{C}_{f.b}$, $K \in \mathcal{I}_f = \mathcal{I}_{f\,cof}$ so $\emptyset \rightarrowtail K$

and $C \twoheadrightarrow *$. Then RE gives $K \triangleright C \twoheadrightarrow K \triangleright * \times_{\emptyset \triangleright *} \emptyset \triangleright C$

As $K \triangleright * = * = \emptyset \triangleright ?$ by Prop 16 3) + 4)

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we s. $K \triangleright C \twoheadrightarrow +$, ie $K \triangleright C \in \mathcal{C}_{f.b.}$

Th $\triangleright$ restricts to

$$(\mathcal{I}_f)^{op} \times \mathcal{C}_{f.b} \longrightarrow \mathcal{C}_{f.b.}$$

This restriction satisfies the right enrichment axioms (81 p 51) dualized as on 85 p 146.

RIa1) holds since ~~$\phi \to K$~~ $\phi \rightarrowtail K$ $\forall K \in \mathcal{I}_f$ by Prop 16 4) and RE

RIa2) holds since $\forall C \in \mathcal{C}_{f.b}$ $C \twoheadrightarrow +$ by Prop 16 3) and RE.

RIb) is empty

RIc) is just RE.

3) and 4) hold since we assume $\mathcal{C}$ is ω-summable and $\aleph$-productive.

~~8.~~ This verifies the weak hypotheses.

The strong hypotheses of ~~#~~ 83 p 143 dualized (cf 85 p 148) just adds to the weak hypothe.

3) = RWN $\exists$ $C \xrightarrow{\sim} \Delta[0] \triangleleft C$, which holds a fortior by ~~Lemma 28~~ a) above

4) = RWNEX, which holds by ~~Lemma 28~~ c) above

and 5) = wES + $\aleph$EP which hold by ~~Lemma 28~~ ~~in~~ d) and b) above.

This verifies the dual of the strong hypotheses.

The extra hypothesis of 85 p 136 dualized, cf 85 p 149, follows from c) above

QED

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Now Let's go back to 85 Th 161.

Suppose now the $\mathcal{C}$ in addition to hypotheses there

is strongly $\aleph$-productive

strongly ω-summable

and satisfies Rich Hypotheses.

85 Th 161 Then gives a model structure

$\mathrm{Cat}(\underline{K}, \mathcal{C})$

Is it Rich Hyp.

Given $F, G : \underline{K} \to \mathcal{C}$.

Its internal hom is

$$\underline{\mathrm{Hom}}_{\mathrm{Cat}(\underline{K}, \mathcal{C})}(F, G) = \int^{\mathrm{end}}_{\underline{K}^{op} \times \underline{K}} (K, K') \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(FK, GK')$$

Note we must take end ~~ex~~ here rather than

hoend since I want

$$\left[\underline{\mathrm{Hom}}_{\mathrm{Cat}(\underline{K}, \mathcal{C})}(F, G)\right]_0 \cong \mathrm{Cat}(\underline{K}, \mathcal{C})(F, G)$$

$$= \int^{\mathrm{end}} (K, K') \to \mathcal{C}(FK, GK')$$

$$= \int^{\mathrm{end}} (K, K') \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(FK, GK').$$

The end will give an enrichment on Δ^{op}-sets

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Best guess for $K\otimes$, $K\triangleleft$ is to form pointwise from $\otimes$, $\triangleleft$ in $\mathcal{C}$.

~~$K\otimes$~~

$L \in \mathcal{L}$, $F \in \mathrm{Cat}(\underline{K}, \mathcal{C})$

$$L \otimes F : K \longmapsto L \otimes FK.$$

$$L \triangleleft F : K \longmapsto L \triangleleft FK.$$

This does have the right variances.

and $\mathrm{Mor}(K, \underline{\mathrm{Hom}}(F, G))$

$= \mathrm{Mor}(K, \int_{\underline{K}}^{\mathrm{end}} \underline{\mathrm{Hom}}_{\mathcal{C}}(FL, GL'))$

$= \int_{\underline{K}}^{\mathrm{end}} \mathrm{Mor}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(FL, GL'))$

$\cong \int_{\underline{K}}^{\mathrm{end}} \underline{\mathrm{Hom}}_{\mathcal{C}}(FL, K \triangleleft GL')$

$= \underline{\mathrm{Hom}}_{\mathrm{Cat}}(F, K \triangleleft G)$

and since $\underline{\mathrm{Hom}}_{\mathcal{C}}(F \otimes K, G)$

$= \underline{\mathrm{Mor}}(K, \underline{\mathrm{Hom}}(F, G))$

But now want to modify definition of fibration

So $G \twoheadrightarrow G'$ is a fibration if

1) $\forall K$ $GK \twoheadrightarrow GK'$ is a fibration (ie its pointwise a f.

2) $\forall$ $F \overset{\sim}{\rightarrowtail} F'$ a equicofibration (defined pointwise)

$$\underline{\mathrm{Hom}}_{\mathrm{Cat}(\underline{K},\mathcal{C})}(F', G) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Cat}(\underline{K},\mathcal{C})}(F, G) \times_{\underline{\mathrm{Hom}}_{\mathrm{Cat}(\underline{K}}(F, G')} \underline{\mathrm{Hom}}(F', G')$$

is a K_a fibration (previous def was surjective on $\Delta[0]$ similarly

---

If $F \rightarrowtail F'$, then $L \rightarrowtail L'$ in $\mathcal{L}$

$$L' \otimes F \sqcup_{L \otimes F} L \otimes F' \rightarrowtail L' \otimes F'$$

as this is pointwise.

Similarly for the rest of $\underline{LE}$.

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So problem is just $\underline{RE}$

But given $K \rightarrowtail L$ and $G \twoheadrightarrow G'$ in $\underline{Cat}(\underline{K}, C)$

$$L \triangleleft G \longrightarrow K \triangleleft G \underset{K \triangleleft G}{\times} L \triangleleft G'$$

so $F \xrightarrow{\sim} F'$

~~Hom~~ (

$$\underline{Hom}(F', L \triangleleft G) \longrightarrow \underline{Hom}(F, L \triangleleft G) \times \underline{Hom}(F', K \triangleleft G \underset{K \triangleleft G}{\times} L \triangleleft G')$$

$$\underline{Hom}(F, K \triangleleft G \underset{K \triangleleft G}{\times} L \triangleleft G')$$

$$\cong$$

$$\underline{Hom}(F, L \triangleleft G) \times \underline{Hom}(F', K \triangleleft G) \times \underline{Hom}(F', L \triangleleft G')$$

$$\left(\underline{Hom}(F, K \triangleleft G) \times \underline{Hom}(F, L \triangleleft G') \right.$$

$$\underline{Hom}(K \triangleleft G)$$

$$\underline{Hom}(F, K \triangleleft G) \times \underline{Hom}(F, L \triangleleft G')$$

$$\underline{Hom}(F, K \triangleleft G')$$

$\cong$

~~Hom(L⊗F')~~

$$\underline{Hom}(L \otimes F', G) \longrightarrow \underline{Hom}(L \otimes F, G) \times \underline{Hom}(K \otimes F', G) \times \underline{Hom}(L \otimes F', G')$$

$$\left(\underline{Hom}(K \otimes F, G) \times \underline{Hom}(L \otimes F', G') \right.$$

$$\underline{Hom}(K \otimes F', G')$$

$$\parallel$$

$$\underline{Hom}(L \otimes F \underset{K \otimes F}{\sqcup} K \otimes F', G) \times \underline{Hom}(L \otimes F', G')$$

$$\underline{Hom}(L \otimes F \underset{K \otimes F}{\sqcup} K \otimes F', G')$$

which is a fibration ~~as G → G'~~ ,,

so $L \otimes F \underset{K \otimes F}{\sqcup} K \otimes F' \xrightarrow{\sim} L \otimes F'$ is a ~~cofib~~ equicofibration

and $G \longrightarrow G'$ satisfies the condition to be a fibration

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Moreover, if $K \xrightarrow{\sim} L \rightarrowtail G \twoheadrightarrow G'$ we ⇒ also a

equimorphism, then since $G \twoheadrightarrow G'$ is also a pointwise fibration

is ▽ ⇒ ~~defined~~ coinitial pointwise

$$L \triangleleft G \xrightarrow{\sim} K \triangleleft G \times_{K \triangleleft G} L \triangleleft G'$$ is a pointwise

equimorphism, and hence an equimorphism

So should be able now to lift enriched shan !

$Cat(\mathbb{K}, \mathcal{C})$.

~~Oh~~ Our problem is need to reprove factorization, as / for §5 : 1.6.1-13

definition of f.s.ation has changed.

But should be able to use holim over general

diagram in $\mathcal{C}$ now.

Then since ⇒ ~~$F \rightarrowtail G$~~ $G \twoheadrightarrow G'$ pointwise fib

$$\begin{array}{ccc} G & \xrightarrow{\sim} & G' \times_{\underset{(\mathbb{K}/?)^{\circ}}{\operatorname{holim}} G'} \operatorname{holim}_{(\mathbb{K}/?)^{\circ}} G \longrightarrow \operatorname{holim} G \\ & \searrow & \downarrow \qquad\qquad \downarrow \\ & & G' \longrightarrow \operatorname{holim} G' \end{array}$$

no,
not unless
$\mathcal{C}$ has all
coequalizers !
~~But~~ ~~has~~ ~~grid~~
~~Fonctor~~ ~~not~~ ~~coreflexive~~
~~transfer~~ ~~functor~~
but old trick of
reduction to fibrant
case still works

a: $F \quad F \xrightarrow{\sim} F'$

$$\underline{Hom}\left(F', \operatorname*{holim}_{(\mathbb{K}/?)^{\circ}} G\right) \longrightarrow \underline{Hom}(F', \operatorname{holim} G') \times_{\underline{Hom}(F, \operatorname{holim} G')} \underline{Hom}(F, \operatorname{holim} G)$$

$$= \int^{\text{end}} = \operatorname{holim}_{\mathbb{K}^{\circ} \times \mathbb{K}} \left(\underline{Hom}_{\mathcal{C}}(Fk', Gk') \longrightarrow \underline{Hom}(\cdots \right.$$

so should be OK

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Will also use $\underline{\mathrm{Hom}}(F \quad)$ measure pullback

to the $\underline{B4}$, etc. by

Suff to s

$\forall\ \Delta[n] \in \Lambda_f$

$$\mathrm{Mor}\left(\Delta[n],\ \underline{\mathrm{Hom}}(F,A) \times_{\underline{\mathrm{Hom}}(F,C)} \underline{\mathrm{Hom}}(F,B)\right)$$

$$= \mathrm{Mor}\left(\Delta[n],\ \underline{\mathrm{Hom}}(F,A) \times_{\mathrm{Mor}(\Delta[n],\ \underline{\mathrm{Hom}}(F,C))} \mathrm{Mor}(\Delta[n],\ \underline{\mathrm{Hom}}(F,B))\right)$$

ie $\mathrm{Mor}_{\mathcal{C}}(\cancel{\mathrm{Hom}}\ F \otimes \Delta[n],\ A) \times_{\mathrm{Mor}_{\mathcal{C}}(F \otimes \Delta[n],\ C)} \mathrm{Mor}_{\mathcal{C}}(F \otimes \Delta[n],\ B)$

$$= \mathrm{Mor}_{\mathcal{C}}\left(F \otimes \Delta[n],\ A \times_C B\right).$$

which is OK by def. of pullback

So looks good

<u>Lemma</u>: Let $\mathcal{C}$ be a category enriched over $\underline{\Delta}^{op}$-sets via $\underline{\mathrm{Hom}} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \underline{\Delta}^{op}$-sets, and suppose for Λ_f = the subcategory of finite simplicial sets there are functors

cf p. 28

$$\Lambda_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

$$\Lambda_f^{op} \times \mathcal{C}^{op} \xrightarrow{\triangleright} \mathcal{C}$$

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with natural isomorphisms

$$\underline{Hom}(C \otimes K, C') \cong \underline{Map}(K, \underline{Hom}(C, C')) \cong \underline{Hom}(C, K \triangleleft C').$$

Then if $\underline{L} \xrightarrow{F} \mathcal{C}$ is a diagram with a limit $\varprojlim F \in \mathcal{C}$

$$\underline{Hom}(C', \varprojlim F) \longrightarrow \varprojlim \underline{Hom}(C', F)$$ is an isomorphism

in $\underline{\Delta^{op}\text{-sets}}$.

Dually if $\underline{L}^{op} \xrightarrow{F} \mathcal{C}$ is a diagram with a colimit $\varinjlim F \in \mathcal{C}$,

then $$\underline{Hom}(\varinjlim F, C') \longrightarrow \varprojlim \underline{Hom}(F, C')$$

is an isomorphism.

Pf: The "dual statement" is indeed the dual under the duality of §15.

So it suffices to prove the first statement.

As limits are formed pointwise, $Cat(\Delta^{op}, Sets) = \underline{\Delta^{op}Set}$,

it suffices to show $\forall n$ that

$$\left(\underline{Hom}(C', \varprojlim F)\right)_n \longrightarrow \varprojlim \underline{Hom}(C', F)_n$$ is an isomorphism

of sets.

But this map identifies to

$$Map_{\mathcal{C}}(C' \otimes \Delta[n], \varprojlim F) \longrightarrow \varprojlim Map_{\mathcal{C}}(C' \otimes \Delta[n], F)$$

which is an isomorphism by definition of $\varprojlim$.

QED

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Prop: Let $\mathcal{C}$ be a left semi-model category (81, p78)

~~Let $\mathcal{N}$ be an infinite cardinal~~

~~Suppose $\mathcal{C}$~~

Suppose arbitrary coequalizers are representable in $\mathcal{C}$.

Suppose $\mathcal{C}$ has coproducts of ~~ob~~ objects over index sets of cardinality $< \mathcal{N}$. (Note then $\mathcal{C}$ has all colimits over ~~diagram~~ diagram of cardinality $< \mathcal{N}$)

Suppose $\mathcal{C}$ satisfies the other conditions to be strongly $\mathcal{N}$-coproductive ~~and strongly ultra ω-cosummable~~

(81, p118, ~~p...~~) and ultra ω-cosummable (p44 below)

Suppose finally that $\mathcal{C}$ is enriched over Δ^{op}-sets, and has a funct

$1_f \otimes \mathcal{C} \longrightarrow \mathcal{C}$ st $\exists$ a natural iso

$\underline{\mathrm{Hom}}(L \otimes C, C') \cong \underline{\mathrm{Map}}(L, \underline{\mathrm{Hom}}(C, C'))$

and $\underline{LE}$ (p14) hold.

~~Suppose there is a funct $1_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ which satisfies the axioms for strong left enrichment (this 86 p2, ... p6)~~

Then if $\underline{K}$ is a category with $< \mathcal{N}$ morphisms:

For any $F : (\Delta/\underline{K})^{op} \longrightarrow \mathcal{C}$

the coend

$$\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F = \int^{\mathrm{coend}}_{\Delta/\underline{K} \times (\Delta/\underline{K})^{op}} (\varphi', \varphi) \longmapsto \Delta[\#\varphi'] \otimes F\varphi$$

$$\cong \int^{\mathrm{coend}}_{\Delta} (p', p) \longmapsto \Delta[p'] \otimes \Big(\bigsqcup_{\varphi : \underline{p} \to \underline{K}} F\varphi \Big)$$

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is representable.

If $\eta : F \longrightarrow F'$ is a natural transformation s.t.

$\forall \varphi \in (\Delta/\underline{K})^{op}$ $F\varphi \longrightarrow F'\varphi$ is a cofibration,

then the induced map

$\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, \eta : \underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F \longrightarrow \underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F$

is a cofibration.

If each $\eta(\varphi) : F\varphi \longrightarrow F\varphi'$ is an equicofibration,

$\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, \eta$ is a equicofibration.

If $\forall \varphi$ $\eta(\varphi) : F\varphi \longrightarrow F\varphi'$ is an equimorphism

AND if $\forall \varphi$ $F\varphi$, $F\varphi'$ are cofibrant,

then $\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, \eta$ is a equimorphism.

of addendum p 49

haven't assumed functorial factorizat. yet. so don't have this set.

Pf: As $\#(\Delta/\underline{K} \times (\Delta/\underline{K})^{op}) < \aleph_1$, and the coend can be expressed as a colimit over a diagram whose cardinality is then $< \aleph_1$ (in fact the ob. $Sd(\underline{K})$, this is somewhere above.

But also, MacLane, Categories for the Working Mathematician

X 5 Prop 1).

Thus the coend over $\Delta/\underline{K}$ exists, ($\aleph_1$-cocompleteness of $\mathcal{C}$)

The isomorphism with the end over Δ is §3 p 158

As on §§2 §3 p

As on §5 p 6-8, if we filter $\Delta/\underline{K}$ by

sieves $sk_n \Delta/\underline{K} =$ $\varphi : \underline{k} \longrightarrow \underline{K}$ st $\exists$ $q \leq n$ st

$\varphi : \underline{k} \longrightarrow \underline{K}$ factors $\underline{k} \longrightarrow \underline{q} \searrow^{\varphi} \underline{K}$

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$(sk_{-1}\,\Delta/K = \emptyset$!

Then

$$\int_{\Delta/K} \Delta[\,] \otimes F \;=\; \varinjlim_{n<\omega} \int_{sk_n \Delta/K} \Delta[\,] \otimes F$$

and $$\int_{sk_{n-1}\Delta/K} \Delta[\,] \otimes F \longrightarrow \int_{sk_n \Delta/K} \Delta[\,] \otimes F$$

f.ts in a pushout square

$$\begin{array}{ccc} \coprod_{\varphi: n \to K \text{ non degenerate}} \dot\Delta[n] \otimes F\varphi & \longrightarrow & \int_{sk_{n-1}\Delta/K} \Delta[\,] \otimes F \\ \downarrow & & \downarrow \\ \coprod_{\varphi: n \to K \text{ non degenerate}} \Delta[n] \otimes F\varphi & \longrightarrow & \int_{sk_n \Delta/K} \Delta[\,] \otimes F \end{array}$$

(but now do not claim left vertical map is cofibration),

This follows from the analysis of factorizat in Δ

(each simplex of Δ/K is degeneracy of a unique non-degenerate simplex)

and the general lemmas §3, 4, 6, and

that $sk_{n-1}(\Delta/K)/\varphi \cong sk_{n-1}\Delta/n$ = colimit of simplices of $\dot\Delta[n]$,

so $\operatorname{colim}_{(sk_{n-1}\Delta/K)/\varphi} (\Delta[\,] \otimes C) \cong (\operatorname{colim}_{sk_{n-1}\Delta/K/\varphi} \Delta) \otimes C \cong \dot\Delta[n]$

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using Prop. ~~17~~ 3) p 17 (proof shows part used need so out $\otimes$, not $\triangleright$)

and ~~Lemma 83 p 68~~

$\operatorname{colim}_{\text{simplice } \dot\Delta[n]} \Delta[\,] \cong \dot\Delta[n]$ (eg. 82 p 5, 68)). Also need trivial $\varinjlim_{\omega} \int_{\mathrm{sk}_n \Delta/\mathbb{E}} \cong$

Clearly if F takes cofibrant values, this all reduces to the $\operatorname{hocofoo}_{\Delta/\mathbb{E}}$ of 85 p 9.

using Prop. 3) p 17 (proof shows the part about $\otimes$ used doesn't need existence of $\triangleright$) and $\operatorname{colim}_{\text{simplice } \dot\Delta[n]} \Delta[\,] \cong \dot\Delta[n]$ (eg 82 p 5, ~~Sh~~ Sublemma p 68).

Also need the trivial $\varinjlim_{n} \int_{\mathrm{sk}_n \Delta/\mathbb{E}} = \int_{\cup \mathrm{sk}_n \Delta/\mathbb{E}} = \int_{\Delta/\mathbb{E}}$.

Note if F take cofibrant values, this $\operatorname{hocofoo}_{\mathbb{E}} F$ is indeed ~~the~~ the $\operatorname{hocofoo}_{\mathbb{E}} F$ of 85 p 9.

It remains to prove the 2 claims on p ~~42~~. 41.

~~The last follows from the first~~

Let $\gamma : F \longrightarrow F'$ be a natural transformation which for all φ is a cofibration (resp, equicofibration) $\gamma(\varphi) : F\varphi \rightarrowtail F'\varphi$.

I ~~claim~~

I need to define:

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Def ultra ω-cosummable Let $\mathcal{C}$ be a left semi-model category.

$\mathcal{C}$ is ultra-ω-cosummable if the following condition holds

i) any sequence $C : \omega \longrightarrow \mathcal{C}$ $\quad C_0 \to C_1 \to C_2 \to \cdots$ has a colimit

ii) Let $f_\bullet : C_\bullet \longrightarrow D_\bullet$ be a map of sequences

$$\begin{array}{ccccccc} C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ D_0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots \end{array}$$

Then if

cf 87 v 117

Def: ultra ω-cosummable Let $\mathcal{C}$ be a left semi-model category.

$\mathcal{C}$ is ultra ω-cosummable if the following conditions hold

i) any tower whatsoever in $\mathcal{C}$, $\quad C_0 \to C_1 \to C_2 \to \cdots$ has a colimit $\varinjlim_{\omega} C_i$

ii) Let $f_\bullet : C_\bullet \longrightarrow D_\bullet$ be a map of towers

$$\begin{array}{ccccccc} C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ D_0 & \longrightarrow & D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots \end{array}$$

suppose that $\forall i : C_i \longrightarrow D_i$ is a cofibration, and that $\forall i : C_{i+1} \underset{C_i}{\sqcup} D_i \longrightarrow D_{i+1}$ is also a cofibration.

(Note the PO exists in $\mathcal{C}$ as $C_i \rightarrowtail D_i$).

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Then $\varinjlim_{\omega} C_i \xrightarrow{f} \varinjlim D_i$ is a cofibration.

If ~~moreover~~ in addition each $f_i : C_i \to D_i$ is an equicofibration,

then $\varinjlim_{\omega} C_i \xrightarrow{f_i} \varinjlim_{\omega} D_i$ is an equicofibration.

Now since $\dot{\Delta}[n] \rightarrowtail \Delta[n]$ is a cofibration in $\mathcal{S}_f$

and $F\varphi \xrightarrow{\eta\varphi} F'\varphi$ is a cofibration (resp. equicofibration) in $\mathcal{C}$,

LE gives that

$$\Delta[n] \otimes F\varphi \sqcup_{\dot{\Delta}[n]\otimes F\varphi} \dot{\Delta}[n] \otimes F'\varphi \longrightarrow \Delta[n] \otimes F'\varphi$$

is a cofibration (resp. equicofibration)

By $\mathcal{N}$-coproductivity

$$\bigsqcup_{\substack{\varphi : \underline{n} \to \mathbb{E} \\ \text{non deg}}} \Delta[n] \otimes F\varphi \sqcup_{\dot{\Delta}[n]\otimes F\varphi} \dot{\Delta}[n] \otimes F'\varphi \longrightarrow \bigsqcup_{\substack{\varphi \\ \text{nd}}} \Delta[n] \otimes F'\varphi$$

$$\cong$$

$$\Big(\bigsqcup_{\substack{\varphi \\ \text{nd}}} \Delta[n] \otimes F\varphi\Big) \sqcup_{\Big(\bigsqcup_{\substack{\varphi \\ \text{nd}}} \dot{\Delta}[n]\otimes F\varphi\Big)} \Big(\bigsqcup_{\substack{\varphi \\ \text{nd}}} \dot{\Delta}[n] \otimes F'\varphi\Big) \longrightarrow \bigsqcup_{\substack{\varphi \\ \text{nd}}} \Delta[n] \otimes F'\varphi$$

is a cofibration (resp. equicofibration).

By ~~LB4~~ we may pushout

~~I show by induction on n that~~

~~η induces a cofibration (resp. equicofibration)~~

$$\cancel{\int_{sk_n \Delta/\mathbb{E}} \Delta[\,] \otimes F \longrightarrow \int_{sk_n \Delta/\mathbb{E}} \Delta[\,] \otimes F'}$$

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~~A fact~~

I will prove by induction on n starting at $n=0$

~~that~~

$$\int_{sk_n \Delta/K} \Delta[\,] \otimes F \bigsqcup_{\int_{sk_{n-1}\partial\Delta/K}} \Delta[\,] \otimes F \int_{sk_{n-1}\Delta/K} \Delta[\,] \otimes F' \longrightarrow \int_{sk_n \Delta/K} \Delta[\,] \otimes F$$

~~is a cofibration~~

and $$\int_{sk_n \Delta/K} \Delta[\,] \otimes F \longrightarrow \int_{sk_n \Delta/K} \Delta[\,] \otimes F'$$

are cofibrations (resp. equicofibrations).

~~For start with $n=0$:~~

~~Proof~~

For $n=-1$, $\int_{sk_{-1}} \Delta[\,] \otimes F \longrightarrow \int_{sk_n} \Delta[\,] \otimes F'$ is $\emptyset \xrightarrow{\sim} \emptyset$.

So assume $n \geq 0$, and ~~that~~ by induction hypothesis that I know

$$\int_{sk_{n-1} \Delta/K} \Delta[\,] \otimes F \longrightarrow \int_{sk_{n-1} \Delta/K} \Delta[\,] \otimes F'$$

is a cofibration (resp. ~~trivial~~ equicofibration)

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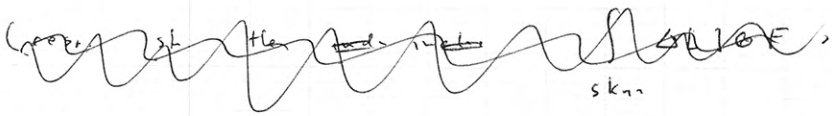
Then LB4 gives a pushout

$$\int_{sk_n} \Delta[\,]\otimes F \;\underset{\int_{sk_{n-1}} \Delta[\,]\otimes F}{\sqcup}\; \int_{sk_{n-1}} \Delta[\,]\otimes F' \quad \text{in } \mathcal{C}$$

~~(resp. st the ... $\int_{sk_{n-1}} \Delta[\,]\otimes F$,~~



and the

$$\int_{sk_n} \Delta[\,]\otimes F \longmapsto \int_{sk_n} \Delta[\,]\otimes F \;\underset{\int_{sk_{n-1}} \Delta[\,]\otimes F}{\sqcup}\; \int_{sk_{n-1}} \Delta[\,]\otimes F'$$

is a cofibration (resp. equicofibration)

Then using the cofibration (resp. equicofibrat) of p45 and the pushout square of p42, I get a pushout square

$$\begin{tikzcd}
\bigsqcup_{\substack{\varphi \\ n-1}} \Delta[\,]\otimes F_\varphi \;\underset{\left(\bigsqcup_{\substack{\varphi \\ n}} \dot\Delta[n]\otimes F_\varphi\right)}{\sqcup}\; \left(\bigsqcup_{\substack{\varphi \\ n-1}} \dot\Delta[n]\otimes F'_\varphi\right) \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "(PO)"] & \bigsqcup_{\substack{\varphi \\ n-1}} \Delta[n]\otimes F'_\varphi \arrow[d] \\
\int_{sk_{n-1}} \Delta[\,]\otimes F \;\underset{\int_{sk_{n-1}} \dot\Delta[\,]\otimes F}{\sqcup}\; \int_{sk_{n-1}} \dot\Delta[\,]\otimes F' \arrow[r] & \int_{sk_n} \Delta[\,]\otimes F
\end{tikzcd}$$

By LB4 the bottom map is a cofibration (resp. trivial cofibration). This gives the first statement be ~~the~~ proved in the induction step. The second follows

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ζ, composition with $\int_{sk_n} \Delta[\,]\otimes F \longrightarrow \int_{sk_n} \Delta[\,]\otimes F \sqcup \int_{sk_{n-1}} \Delta[\,]\otimes F'$

$\int_{sk_{n-1}} \Delta[\,]\otimes F$

~~But~~ This concludes the induction step.

But now, since $\forall n \geq 0$

$$\int_{sk_n \Delta/K} \Delta[\,]\otimes F \rightarrowtail \int_{sk_n \Delta/K} \Delta[\,]\otimes F'$$

ζ

and $\int_{sk_{n+1}\Delta/K} \Delta[\,]\otimes F \sqcup_{\int_{sk_n \Delta/K} \Delta[\,]\otimes F} \int_{sk_n \Delta/K} \Delta[\,]\otimes F' \longrightarrow \int_{sk_{n+1}\Delta/K} \Delta[\,]\otimes F'$

are cofibrations (resp. equicofibrations) ultra ω-cosaturnos.t.

gives that

$$\varinjlim_n \int_{sk_n \Delta/K} \Delta[\,]\otimes F \longrightarrow \varinjlim_n \int_{sk_n \Delta/K} \Delta[\,]\otimes F'$$

$$\int_{\Delta/K} \Delta[\,]\otimes F \qquad \int_{\Delta/K} \Delta[\,]\otimes F'$$

is a cofibration (resp. equicofibration) as required.

QED.

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Addendum: If in addition to the hypotheses of Prop. 40,

$\mathcal{C}$ satisfies left functorial factorization (LFF 83, 93 cf 81, 49)

then, for any ~~weak equivalence~~

natural $\gamma: F \longrightarrow F'$ between functor $\Delta/\underline{K} \longrightarrow \mathcal{C}$

st 1) $\forall \varphi$ $\gamma\varphi: F\varphi \longrightarrow F\varphi'$ is an equimorphism ~~AND~~

~~2) $\forall \varphi$, $F\varphi$, $F\varphi'$ are cofibrant~~, between cofibrant objects

then $\underset{\Delta/\underline{K}}{\text{hocofoo}}\ \gamma: \underset{\Delta/\underline{K}}{\text{hocofoo}}\ F \longrightarrow \underset{\Delta/\underline{K}}{\text{hocofoo}}\ F'$

is an equimorphism

Pf: Follows from Prop. 40 by method of 85, p 12.

Or use 85 Lemma 115 and 85 Prop 32, and Prop 40.

Or note for $F \in \mathrm{Cat}(\Delta/\underline{K}, \mathcal{C}_{cof})$

the $\underset{\Delta/\underline{K}}{\text{hocofoo}}\ F$ reduces to that of 85 Prop 9, for

which the assertion holds. (85 Prop. 9 b), p 11).

Remark: $\mathcal{C} = \Delta^{or}\mathrm{Sets}$ with its usual model structure

satisfies all the hypotheses of Prop 40.

It also satisfies the dual hypotheses:

to check ultra ω-summability use method similarly

to 85 p 176–177

For given towers in Δ^{or}-set

$$\begin{array}{ccccccc} \to X_n & \to & X_{n-1} & \to \cdots & \to & X_0 & \to \cdot \\ \downarrow f_n & & \downarrow f_{n-1} & & & \downarrow f_0 & \\ \to Y_n & \to & Y_{n-1} & \to \cdots & \to & Y_0 & \to \cdot \end{array}$$

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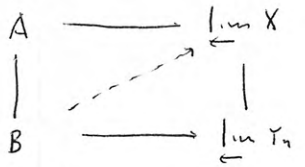
With $f_n : X_n \to Y_n$ and $X_{n+1} \to Y_{n+1} \times_{Y_n} X_n$ Kan fibrations (resp. ~~equivalences~~ weak equivalences and Kan fibrations)

the $\varprojlim X_n \to \varprojlim Y_n$ ~~is~~ is a Kan fibration

(resp. weak equivalence also Kan fibration).

~~If~~ For it suffices to check ~~for~~ the existence of a

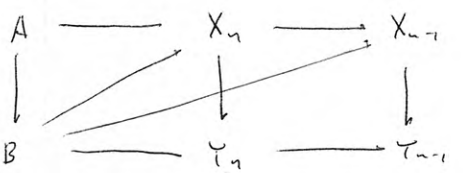
fill-in for any diagram ~~of~~ of the form



Where $A \xrightarrow{\sim} B$ is a equivalence (resp. a cofibration).

For the B, universal property of $\varprojlim$ it suffices to find

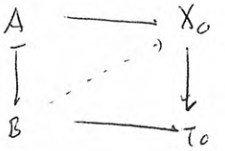
Compatible fillings :



$\forall n$.

But we construct those ξ_n inductive in n.

For $n = 0$,



such $A \xrightarrow{\sim} B$ (resp. $X_0 \xrightarrow{\sim} Y_0$)

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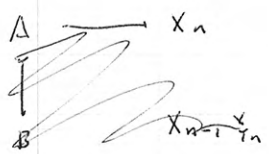
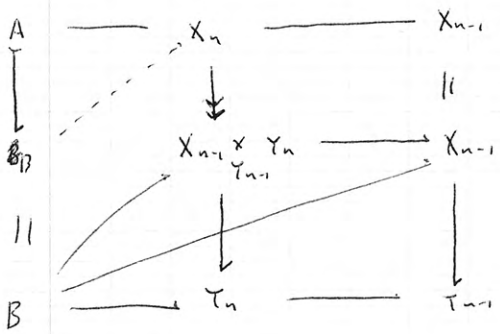
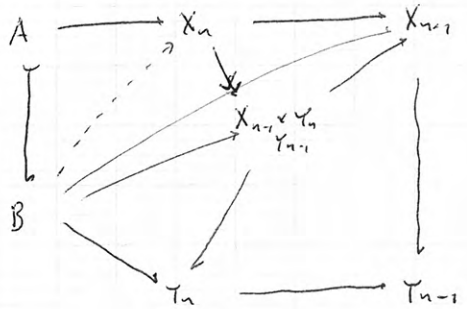
Given $B \longrightarrow X_{n-1}$ we set







$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X_n \\ B \arrow[ur, dashed] & X_{n-1} \times_{Y_n} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X_n \arrow[r] \arrow[d, two heads] & X_{n-1} \arrow[d, equal] \\ B \arrow[ur, dashed] \arrow[d, equal] & X_{n-1} \times_{Y_{n-1}} Y_n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ B \arrow[r] \arrow[ur] \arrow[urr] & Y_n \arrow[r] & Y_{n-1} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & X_n \arrow[r] \arrow[d, two heads] & X_{n-1} \arrow[dd] \\ B \arrow[ur, dashed] \arrow[urr] \arrow[dr] & X_{n-1} \times_{Y_{n-1}} Y_n \arrow[ur] \arrow[dl] & \\ & Y_n \arrow[r] & Y_{n-1} \end{tikzcd}$$

But then as $X_n \twoheadrightarrow X_{n-1} \times_{Y_{n-1}} Y_n$ is a fibration,

and either it or $A \rightarrowtail B$ is ~~an eq~~ a weak equivalence,

we have a lift $B \dashrightarrow X_n$ as indicated.

This completes the ~~[illegible]~~ inductive step of the construction QED

Lemma

~~Prop.~~ Let $\mathcal{C}$ be a category enriched over Δ^{op}-Sets,

with ~~[illegible]~~ $\mathcal{1}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ $\quad \mathcal{1}_f^{op} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$

s.t. $\exists$ natural isomorphisms $\forall L \in \mathcal{1}_f$, $\forall C, C' \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(L \otimes C, C') \cong \underline{\mathrm{Map}}(L, \underline{\mathrm{Hom}}(C, C')) \cong \underline{\mathrm{Hom}}(C, L \triangleright C')$$

Suppose $\mathcal{C}$ is a left model category with functorial factorization ~~[illegible]~~ (81 p 48, 83 p 93) and that $\mathcal{C}$ satisfies $\lambda\backslash\mathcal{L}$-coproductivity and ω-cosummability (81 p 118, 117)

Suppose that the enrichment satisfies the axiom LE of p 28

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a) The $\mathcal{C}$ ~~satisfies~~ satisfies:

the weak hypotheses of 81, 143

the strong hypotheses of 83, 143

the extra hypotheses of 85, 136

~~b) For any $F : \underline{K}$~~

If $\# \underline{K} < \aleph$, for any $F : (\Delta/\underline{K})^{op} \longrightarrow \mathcal{C}_{cof}$

and any $G : (\Delta/\underline{K})^{op} \longrightarrow \mathcal{C}$

there is a natural isomorph

$$\underline{\mathrm{Hom}}(\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F, G) \cong \underset{\underline{K}}{\mathrm{hoend}}\left(K, K' \longmapsto \underline{\mathrm{Hom}}(FK, GK')\right)$$

cf 85 Thm 140

c) ~~If $\underline{L} \times \underline{K} \xrightarrow{F} \mathcal{C}$ is any diagram~~

If $\Phi : \underline{L} \longrightarrow \mathrm{Cat}(\underline{K}, \mathcal{C}_{cof})$ is any

* diagram of functors $F_L : \underline{K} \longrightarrow \mathcal{C}_{cof}$

s.t. $\varinjlim_{\underline{L}} F_L : \underline{K} \longrightarrow \mathcal{C}_{cof}$ exists,

then $\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\left(\varinjlim_{\underline{L}} F_L\right)$ represents $\varinjlim_{\underline{L}}\left(L \to \underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F_L\right)$

Pf:

a) follows by the dual to the proof of Lemma 28:

We note this ~~usage~~ uses only one side of the model structure.

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~~b) Given a), b) follows as in proof of Thm 140 85 Th p140. Note since the hoend is taken in Δ^{or}-sets, we don't need it to be Hom~~

b) Given a), b) follows as in the proof of 85 Thm 140 b), stripped of unneed hypotheses.

Namely we natural natural isos $\forall n \in \Delta$

$$\underline{\mathrm{Hom}}(\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F, G)_n = \mathrm{Mor}_{\mathcal{C}}(\Delta[n] \otimes \underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, F, G)$$

$\otimes$ is like cotensor for $\underline{\mathrm{Hom}}$ ("like" as $\otimes$ defines only on $1_f \in \Delta^{or}$-sets

$$\cong \mathrm{Mor}_{\mathcal{C}}(\underset{\Delta/\underline{K}}{\mathrm{hocofoo}}\, (\Delta[n] \otimes F), G)$$

85 Sublemma p75

$$\cong \underset{\underline{K}}{\mathrm{hoend}}\,(K, K' \longmapsto \mathrm{Mor}_{\mathcal{C}}(\Delta[n] \otimes FK, GK'))$$

~~85 p123 b)~~

$$\cong \underset{\underline{K}}{\mathrm{hoend}}\,(K, K' \longmapsto \underline{\mathrm{Hom}}(FK, GK')_n)$$

$$\cong \left(\underset{\underline{K}}{\mathrm{hoend}}\,(K, K' \longmapsto \underline{\mathrm{Hom}}(FK, GK'))\right)_n$$

limits in Δ^{or}sets formed pointwise on Δ.

As this iso is natural in $n \in \Delta$, it gives an iso of simplicial sets as required.

(Note: 85 Thm 140 asked the ~~Hom(FK, GK)~~ G to take fibrant values so $\underline{\mathrm{Hom}}(FK, GK')$ would be a Kan complex. But we don't need this, hoends (even hofoo's) exists for arbitrary diagrams of Δ^{or}-sets (cf Remark p49). Then we don't need a $\mathcal{C}$f.s, so only need the left model structure of $\mathcal{C}$. This means hypotheses of 85 140 can be discarded.).

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c). For any $X \in \mathcal{C}$

$$\mathrm{Mor}_{\mathcal{C}}\left(\underset{\Delta/\underline{K}}{\mathrm{hocofoo}} \left(\varinjlim_{\underline{K}} F_L \right), \quad X \right)$$

$$= \underset{\underline{K}}{\mathrm{hoend}} \left(K, K' \right.$$

cf 85 p140,

Prop. Let $\mathcal{C}$ be a category enriched over Δ^{op}-sets with $\mathcal{A}_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ and $\mathcal{A}_f^{op} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$ s.t. $\exists$ natural isos $\forall$ $L \in \mathcal{A}_f$, $C, C' \in \mathcal{C}$

$$\underline{\mathrm{Hom}}(L \otimes C, C') \cong \underline{\mathrm{Map}}(L, \underline{\mathrm{Hom}}(C, C')) \cong \underline{\mathrm{Hom}}(C, L \triangleright C')$$

Suppose $\mathcal{C}$ is a left model category with functorial factorization (81 p48, 83 p93) which satisfies $\aleph$-coproductivity and ω-cosummability. (81 p118, 117).

Suppose further the enrichment satisfies 2E of p28.

Supp $\#$ mor $\underline{K} < \aleph$.

Then

a) For any $F : (\Delta/\underline{K})^{op} \longrightarrow \mathcal{C}_{cof}$

and any $C' \in \mathcal{C}$

notation

cf 85 p154-157

$$\underline{\mathrm{Hom}}\left(\underset{\Delta/\underline{K}}{\mathrm{hocofoo}} F, C' \right) \cong \underset{(\Delta/\underline{K})^{op}}{\mathrm{hofoo}} \left(\varphi \longmapsto \underline{\mathrm{Hom}}(F\varphi, C') \right)$$

b) For any $F : \underline{K} \longrightarrow \mathcal{C}_{cof}$

and any $G : \underline{K} \longrightarrow \mathcal{C}$

there is a natural isomorphism

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$$\int_{\underline{K}}^{\text{end}} (K, K') \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(\operatorname{hocolim}_{\underline{K}/K} F, GK')$$

$$\cong \quad \operatorname{hoend}_{\underline{K}} ((K, K') \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(FK, GK')$$

hoend a special case of hoBaum

c) If $\Phi : \underline{L} \longrightarrow \mathrm{Cat}((\Delta/\underline{K})^{op}, \mathcal{C}_{cof})$

is any diagram of functors s.t.

$\varinjlim_{\underline{L}} \Phi L$ exists as a functor $(\Delta/\underline{K})^{op} \longrightarrow \mathcal{C}_{cof}$

(not it is assumed to take cofibrant values!).

Then $\operatorname{hocofoo}_{\Delta/\underline{K}} (\varinjlim_{\underline{L}} \Phi L)$ in $\mathcal{C}$ represents

$$\varinjlim_{\underline{L}} (L \longrightarrow \operatorname{hocofoo}_{\Delta/\underline{K}} \Phi L).$$

Pf:

Notes By Lemma 51, $\mathcal{C}$ satisfies the weak hypotheses, strong hypotheses, and extra hypotheses. Thus $\operatorname{hocofoo}_{\Delta/\underline{K}} F$ exists by 85 Prop 9.

$\operatorname{hofoo}_{(\Delta/\underline{K})^{op}}$ exists in Δ^{op}-sets for any functor, without need to assume fibrant values. (Prop. 740 + Remark 49)

So both sides exist.

Then ~~natura~~ $\exists$ natural isomorphism in $F, \underline{K}, C'$ a $n \in \Delta$

$$\underline{\mathrm{Hom}}(\operatorname{hocofoo}_{\Delta/\underline{K}} F, C')_n \cong \mathrm{Mor}_{\mathcal{C}}(\Delta[n] \otimes \operatorname{hocofoo}_{\Delta/\underline{K}} F, C')$$

$\otimes$ the partial cotensor

$$\cong \cancel{\mathrm{Mor}_{\mathcal{C}}(\operatorname{hocofoo}_{\Delta/\underline{K}} \Delta}$$

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$$\cancel{\cong \mathrm{Mor}_e(\underset{\Delta/\underline{k}}{\mathrm{hocofoo}}\ \Delta[n]}$$

$$\cong \mathrm{Mor}_e(\underset{\Delta/\underline{k}}{\mathrm{hocofoo}}\ \Delta[n]\otimes F,\ C')$$ [85 ~~Lemma~~ Sublemma p 75

$$\cong \mathrm{Mor}_e\Big(\int^{\mathrm{coend}}_{\Delta/\underline{k}\times(\Delta/\underline{k})^{op}} \Delta[\,]\otimes(\Delta[n]\otimes F),\ C'\Big)$$ [def of hocofoo

$$= \int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \mathrm{Mor}_e(\Delta[\,]\otimes(\Delta[n]\otimes F),\ C')$$ [& universal property of coen

$$\cong \int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \mathrm{Mor}_e(\cancel{\Delta[n]}(\Delta[\,]\times\Delta[n])\otimes F,\ C')$$

$$\cong \int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \mathrm{Mor}_e(\Delta[n]\otimes(\Delta[\,]\otimes F),\ C')$$

enriched + part ~~semi~~cokern

$$\cong \int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \underline{\mathrm{Hom}}(\Delta[\,]\otimes F,\ C')_n$$

$$\cong \left[\int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \underline{\mathrm{Hom}}(\Delta[\,]\otimes F,\ C')\right]_n$$ ends ptwise in Δ^{op}-sets

$$\cong \left[\int^{\mathrm{end}}_{(\Delta/\underline{k})^{op}\times\Delta/\underline{k}} \underline{\mathrm{Map}}(\Delta[\,],\ \underline{\mathrm{Hom}}(F,C'))\right]_n$$

$$= \left[\underset{(\Delta/\underline{k})^{op}}{\mathrm{hofoo}}\ \underline{\mathrm{Hom}}(F,C')\right]_n$$ def of hofoo of 85, p154, 157 and thus 86, p40.

As natural in n, this gives a natural iso of simplicial sets

$$\underline{\mathrm{Hom}}(\underset{\Delta/\underline{k}}{\mathrm{hocofoo}}\ F,\ C') \cong \underset{(\Delta/\underline{k})^{op}}{\mathrm{hofoo}}\ \underline{\mathrm{Hom}}(F,C')$$

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This proves a)

b) Ends exist in the complete category Δ^n-sets, as do hoends of arbitrary functors (cf Prop 40 + Remark 79).

~~end~~

There is a isomorphism natural in $\underline{\underline{K}}, K, K', F, G$ and n.

$$\left[\int_{\underline{\underline{K}}}^{\text{end}} (K,K') \longmapsto \underline{\text{Hom}}_{\mathcal{E}} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/K} F, GK' \right) \right]_n \cong \int_{\underline{\underline{K}}}^{\text{end}} (K,K') \longmapsto \left[\underline{\text{Hom}}_{\mathcal{E}} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/K} F, GK' \right) \right]_n$$

$$\cong \int_{\underline{\underline{K}}}^{\text{end}} (K,K') \longmapsto \underline{\text{Mor}}_{\mathcal{E}} \left(\Delta[n] \otimes \operatorname*{hocolim}_{\underline{\underline{K}}/K} F, GK' \right)$$

$$\cong \int_{\underline{\underline{K}}}^{\text{end}} (K,K') \longmapsto \text{Mor}_{\mathcal{E}} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/K} F \otimes \Delta[n], GK' \right)$$

[suslemm 85 r 75 / cond simpl.]

$$\cong \left[\operatorname*{hoend}_{\underline{\underline{K}}} (K,K' \longmapsto \underline{\text{Hom}}_{\mathcal{E}}(FK, GK') \right]_n$$

$$\cong \text{Mor}_{\text{Cat}(\Delta/\underline{\underline{K}}, \mathcal{E})} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/?} (F \otimes \Delta[?]), G \right)$$

[definition of natural transformation]

$$\cong \text{Mor}_{\text{Cat}((\Delta/\underline{\underline{K}})^{\text{op}}, \mathcal{E})} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/?} (F \otimes \Delta[?]), G \right)$$

$$\cong \text{Mor}_{\text{Cat}(\underline{\underline{K}}, \mathcal{E})} \left(\operatorname*{hocolim}_{\underline{\underline{K}}/?} F \otimes \Delta[?], G \right)$$

[85 r 127]

$$\cong \left[\operatorname*{hoend}_{\underline{\underline{K}}^{\text{op}}} (K,K' \longmapsto \underline{\text{Hom}}_{\mathcal{E}}(F \otimes \Delta[n], G) \right]_0$$

$$\cong \left[\operatorname*{hoend}_{\underline{\underline{K}}^{\text{op}}} (K,K') \longmapsto \text{Mor}(\Delta[n], \underline{\text{Hom}}_{\mathcal{E}}(F,G)) \right]_0$$

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$$\cong \cancel{\underline{\mathrm{Map}}_{\Delta^{op}\mathrm{Set}}(\Delta[n], (\underset{\underline{K}}{\mathrm{hoend}}\ K, K' \to \underline{\mathrm{Hom}}_{\mathcal{C}}(FK, GK'))}$$

~~dual of 88 p 75 in Δ^{op}-Set~~

But in Δ^{op} Set $\underline{\mathrm{Map}}(\Delta[n], \quad)$ preserves all limits or, hence all ends. Thus

$$\underset{\underline{K}}{\mathrm{hoend}}\ (K, K' \longrightarrow \underline{\mathrm{Map}}(\Delta[n], \underline{\mathrm{Hom}}(FK, GK'))$$

cf 87 Sublem p75

$$= \cancel{\int_{(\Delta/\underline{K})^{op}} \underline{\mathrm{Map}}(\Delta[\ ]},$$

$$\int_{(\Delta/\underline{K})^{op}} (\varphi', \varphi) \longmapsto \underline{\mathrm{Map}}(\Delta[\varphi'], \underline{\mathrm{Map}}(\Delta[n], \underline{\mathrm{Hom}}(F\varphi 0, G\varphi(\max)))$$

$$\cong \int^{en}_{(\Delta/\underline{K})^{op}} (\varphi', \varphi) \longmapsto \underline{\mathrm{Map}}(\Delta[\varphi'] \times \Delta[n], \underline{\mathrm{Hom}}(F\varphi 0, G\varphi(\max))$$

$$\cong \int^{end}_{(\Delta/\underline{K})^{op}} (\varphi', \varphi) \longmapsto \underline{\mathrm{Map}}(\Delta[n], \underline{\mathrm{Map}}(\Delta[\varphi'], \underline{\mathrm{Hom}}(F\varphi 0, G\varphi(\max))$$

$$\cong \underline{\mathrm{Map}}(\Delta[n]; \int^{end}_{(\Delta/\underline{K})^{op}} (\varphi', \varphi) \longmapsto \underline{\mathrm{Map}}(\Delta[\varphi'], \underline{\mathrm{Hom}}(F\varphi 0, G\varphi(\max)))$$

$$= \underline{\mathrm{Map}}(\Delta[n], \underset{\underline{K}}{\mathrm{hoend}}\ \underline{\mathrm{Hom}}_{\mathcal{C}}(F, G))$$

So $\left[\underset{\underline{K}^{op}}{\mathrm{hoend}}\ (K, K') \longmapsto \underline{\mathrm{Map}}(\Delta[n], \underline{\mathrm{Hom}}(FK, GK'))\right]_0$

$$\cong \left[\underline{\mathrm{Map}}(\Delta[n], \underset{\underline{K}^{op}}{\mathrm{hoend}}\ \underline{\mathrm{Hom}}_{\mathcal{C}}(F, G))\right]_0$$

$$= \left[\underset{\underline{K}}{\mathrm{hoend}}\ \underline{\mathrm{Hom}}_{\mathcal{C}^{op}}(F, G)\right]_n.$$

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As this isomorphism is natural in n, it gives us the desired natural isomorphism of simplicial ~~[illegible]~~ sets.

$$\int_{\underline{K}}^{\mathrm{end}} (K, K') \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(\operatorname*{hocolim}_{\underline{K}/K} F, G)$$

$$\cong \operatorname*{hoend}_{\underline{K}} (K, K' \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(FK, GK')).$$

This proves b).

~~c) For any $C \in \mathcal{C}$~~

~~$\mathrm{Mor}_{\mathcal{C}}(\operatorname*{hofoo}_{\Delta/}$~~

~~$\mathrm{Mor}_{\mathcal{C}}(\operatorname*{hocofoo}_{\Delta/\underline{K}} F, C)$~~

~~$\mathrm{Mor}_{\mathcal{C}}(\operatorname*{hocofoo}_{\Delta/\underline{K}} (\varinjlim_{\underline{L}} \Phi L), C)$~~

~~$= \left[\operatorname*{hofoo}_{(\Delta/\underline{K})^{op}} (\varphi \longmapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(\varinjlim_{\underline{L}} \Phi L(\varphi), C))\right]_0$ by a)~~

~~$= \operatorname*{hofoo}_{\Delta/\underline{K}^{op}} (\varphi \longmapsto \varprojlim_{\underline{L}^{op}} \mathrm{Hom}$~~

c) ~~$\operatorname*{hofoo}_{\Delta}$~~ $\operatorname*{hocofoo}_{\Delta/\underline{K}} (\varinjlim_{\underline{L}} \Phi L)$, $\operatorname*{hocofoo}_{\Delta/\underline{K}} \Phi L$ exist a

$\varinjlim_{\underline{L}} \Phi L$, $\Phi L : \Delta/\underline{K} \longrightarrow$ ~~[illegible]~~ $\mathcal{C}_{coh}$

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Then

$$\operatorname*{hocofoo}_{\Delta/\underline{\mathbb{E}}} \left(\varinjlim_{\underline{L}} \Phi L \right)$$

$$= \int_{\Delta/\underline{\mathbb{E}}}^{\text{coen}} (\varphi', \varphi) \longmapsto \Delta[\#\varphi'] \otimes \left(\varinjlim_{\underline{L}} \Phi L \right)$$

$$\cong \int_{\Delta/\underline{\mathbb{E}}}^{\text{coen}} (\varphi', \varphi) \longmapsto \varinjlim_{\underline{L}} \left(\Delta[\#\varphi'] \otimes \Phi L \right)$$

$$\cong \varinjlim_{\underline{L}} \int_{\Delta/\underline{\mathbb{E}}}^{\text{coen}} (\varphi', \varphi) \longmapsto \Delta[\#\varphi'] \otimes \Phi L$$

$$= \varinjlim_{\underline{L}} \left(L \to \operatorname*{hocofoo}_{\Delta/\underline{\mathbb{E}}} \Phi L \right).$$

QED.

argument p16 4) ~~made~~ ~~of~~ ~~made~~ one-side

ons needs a left model structure, but enrichment on both sides.

colimits commute

[~~includes representation~~ ~~of such~~ !]

[includes representability statements

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Thm: Let $(\mathcal{C}, \mathrm{cof}\mathcal{C}, \mathrm{equi}\mathcal{C}, \mathrm{fis}\mathcal{C})$ satisfy the basic model axioms (78 156–157) B1–B6, and functorial factorization F1, F'1 (78, 158) (These imply B7. Given richness, will follow from C1 of 79, 172).

Suppose $\mathcal{C}$ is enriched over Δ^{or}-sets, and that there are "partial" tensor and cotensor

$$1_f \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$
$$1_f^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\triangleright} \mathcal{C}$$

($1_f \in \Delta^{\mathrm{or}}\text{-sets}$ = full subcategory of finite Δ^{or}-sets.

s.t.

$$\mathrm{Hom}(X \otimes C, C') \cong \mathrm{Map}(X, \mathrm{Hom}_{\mathcal{C}}(C, C')) \cong \mathrm{Hom}_{\mathcal{C}}(C, X \triangleright C')$$

Suppose axioms RE and LE hold (p. 14, 15).
Suppose that $\mathcal{C}$ satisfies $\aleph_1$-productivity and ω-summasivity (81, 118, 117).

Let $\mathbb{K}$ be a category with $\#\mathrm{mor}\,\mathbb{K} < \aleph_1$.

Used $\mathbb{K}^{\mathrm{or}}$ in 85 p. 161 but no where else.

Then $\mathrm{Cat}(\mathbb{K}, \mathcal{C})$ has a basic model structure with functorial factorization, $\mathcal{C}$ is enriched over Δ^{or}-sets with $\otimes$, $\triangleright$ s.t. LE + RE hold, and $\mathrm{Cat}(\mathbb{K}, \mathcal{C})$ satisfies $\aleph_1$-productivity and ω-summasivity.

This structure is given by

$\gamma: F \to G$ a natural transformation is

i) a cofibration if $\forall K \in \mathbb{K}$, $\gamma K: FK \to GK$ is a cofibration in $\mathcal{C}$

ii) an equimorphism if $\forall K \in \mathbb{K}$, $\gamma K: FK \to GK$ is an equimorphism in $\mathcal{C}$

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iii) a fibration if $\forall K \in \underline{K}$ $FK \longrightarrow GK$ is a fibration in $\mathcal{C}$ and for all equicofibration $A \xrightarrow{\sim} B$ in $Cat(\underline{K}, \mathcal{C})$

$$\underline{Hom}_{Cat(\underline{K},\mathcal{C})}(B, F) \longrightarrow \underline{Hom}_{Cat(\underline{K},\mathcal{C})}(A, F) \times \underline{Hom}(B, G)$$
$$\underline{Hom}(A, G)$$

is a Kan fibration and weak equivalence of simplicial sets.

a) For $X \in \mathcal{S}_f$ and $F : \underline{K} \longrightarrow \mathcal{C}$

$X \otimes F : \underline{K} \longrightarrow \mathcal{C}$ sends $K \longmapsto X \otimes FK$.

$X \triangleleft F$ ~~sends~~ $: \underline{K} \longrightarrow \mathcal{C}$ sends $K \longmapsto X \triangleleft FK$.

b) For $F, G \in Cat(\underline{K}, \mathcal{C})$.

$$\underline{Hom}_{Cat(\underline{K},\mathcal{C})}(F, G) = \int_{\underline{K}}^{end} (K, K') \longmapsto \underline{Hom}_{\mathcal{C}}(FK, GK')$$

c) A natural transformation $\gamma : F \longrightarrow G$ of functors $\underline{K} \longrightarrow \mathcal{C}$ is a cofibration if it is pointwise a cofibration in $\mathcal{C}$ ($\forall K \in \underline{K}$, $\gamma K : FK \longrightarrow GK$ in $\mathcal{C}$).

~~If~~ γ is a equimorphism if it is so pointwise ($\forall K \in \underline{K}$, $\gamma K : FK \xrightarrow{\sim} GK$ is in $\mathcal{C}$.

γ is a fibration if it is pointwise a fibration and for all diagrams

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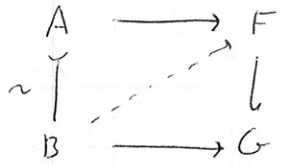

where $A \rightarrowtail B$ is a trivial cofibration, then

$\exists$ a l.s.t.

Pf: By the dual to p 51, $\mathcal{C}$ satisfies the dual weak hypotheses (§1 p 143 dualized), the dual strong hypotheses (§3 p 173, dualized) and the dual extra hypotheses of §3 (§5, 136). These give JXCEP (§5, 149) & RWN (§5, 149), since these are among the dual strong hypotheses.

By §5 Thm p 161, $Cat(\underline{K}^{2n}, \mathcal{C})$ with the described structure is a basic model category with functorial factorization, which is JXC-productive and ω-summable.

Moreover $\underline{Hom}(F,G) = \int^{(K,K') \in \underline{K}^{2n}} (K,K') \longmapsto \underline{Hom}_{\mathcal{C}}(FK, GK')$

is easily seen to give an enrichment over Δ^{cc}-set.

(Note $\left[\underline{Hom}(F,G)\right]_0 = \left[\int^{(K,K') \in \underline{K}^{2n}} (K,K') \longmapsto \underline{Hom}_{\mathcal{C}}(FK, GK')\right]_0$

$= \int^{(K,K') \in \underline{K}^{2n+1}} \left[(K,K') \longmapsto \underline{Hom}_{\mathcal{C}}(FK, GK')\right]$

$= \int^{(K,K') \in \underline{K}^{2n}} (K,K') \longmapsto Mor_{\mathcal{C}}(FK, GK'))$

$= Mor_{Cat(\underline{K},\mathcal{C})}(F,G).$

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Moreover if $X \in \mathcal{S}_f$

$$\underline{\mathrm{Hom}}(X \otimes F, G)$$

$$= \int^{\mathrm{end}} (K, K') \vdash \underline{\mathrm{Hom}}_e(X \otimes FK, GK')$$

$$= \int^{\mathrm{end}} (K, K') \vdash \underline{\mathrm{Map}}(X, \underline{\mathrm{Hom}}_e(FK, GK'))$$

$$\cong \underline{\mathrm{Map}}\left(X, \int^{\mathrm{end}} (K, K') \vdash \underline{\mathrm{Hom}}_e(FK, GK')\right)$$

$$\cong \underline{\mathrm{Map}}(X, \underline{\mathrm{Hom}}(F, G))$$

$$\cong \int^{\mathrm{end}} (K, K') \vdash \underline{\mathrm{Map}}(X, \underline{\mathrm{Hom}}(FK, GK'))$$

$$= \int^{\mathrm{end}} (K, K') \vdash \underline{\mathrm{Hom}}_e(FK, X \triangleright GK'))$$

$$\cong \underline{\mathrm{Hom}}(F, X \triangleright G).$$

It remains only to verify the axiom $\underline{LE}$ and $\underline{RE}$

~~Hold~~ hold.

But given $X \rightarrowtail Y$ in $\mathcal{S}_f$ and $F \rightarrowtail G$. $\mathrm{Cat}(\underline{K}, \mathcal{C})$

$$Y \otimes F \sqcup_{X \otimes F} X \otimes G \longrightarrow Y \otimes G$$ is indeed represented

by a cofibration, since this may be determined

pointwise, and $\forall K \in \underline{K}$

$$Y \otimes FK \sqcup_{X \otimes FK} X \otimes GK \rightarrowtail Y \otimes GK$$ is represented

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by a cofibration in $\mathcal{C}$ by $LE \subset \mathcal{C}$.

Now to check RE.

Let X

If either $X \rightarrowtail Y$ or $F \rightarrowtail G$ is an equivalence,

so is $Y \otimes F \amalg_{X \otimes F} X \otimes G \longrightarrow Y \otimes G$, by LE in $\mathcal{C}$,

this condition being verifiable pointwise.

Now $\subset$ RE.

Let $X \rightarrowtail Y$ be in and $F \longrightarrow G$ be a fibration.

Then $X \triangleright F \longrightarrow X \triangleright G$ ×

The $Y \triangleright F \longrightarrow X \triangleright F \times_{X \triangleright X} Y \triangleright G$

is representable by a pointwise fibration, which is

moreover an equivalence if either $X \xrightarrow{\sim} Y$ or $F \xrightarrow{\sim} G$.

It remains only to show $\subset$ $X \rightarrowtail Y$ or $F \twoheadrightarrow G$

that the pointwise fibration

$$Y \triangleright F \longrightarrow X \triangleright F \times_{X \triangleright G} Y \triangleright G \quad \text{is a fibration,}$$

ie that it also satisfies the lifting property of

the definition of fibration.

But let $A \xrightarrow{\sim} B$ be an equicofibration,

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Note there is an adjunct of $Y, X \in \mathcal{A}_f$

$$X \otimes \dashv X \triangleright$$

For $\underline{\mathrm{Hom}}(X \otimes A, F) \cong \underline{\mathrm{Map}}(X, \underline{\mathrm{Hom}}(A, F)) \cong \underline{\mathrm{Hom}}(A, X \triangleright F)$

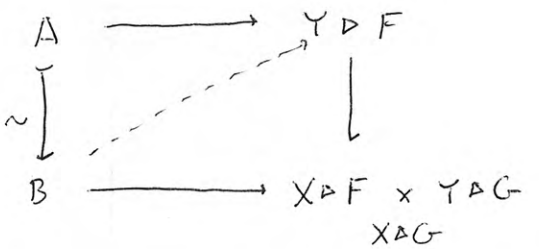
so taking $[\ ]_0$

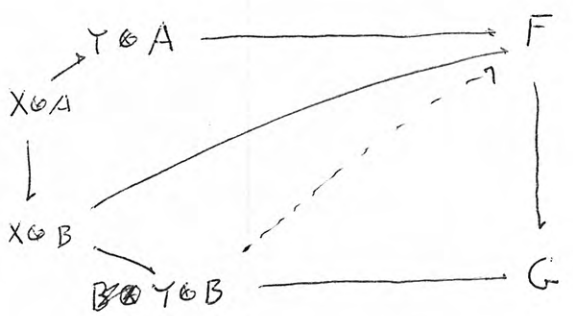
$$\mathrm{Mor}(X \otimes A, F) \cong \mathrm{Mor}(A, X \triangleright F).$$

Then

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "\sim"'] & Y \triangleright F \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & X \triangleright F \times_{X \triangleright G} Y \triangleright G
\end{tikzcd}$$

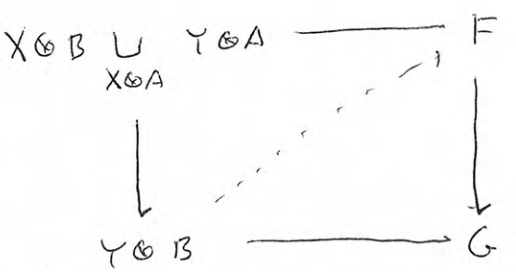
correspond bijectively to





$$\begin{tikzcd}
 & Y \otimes A \arrow[rr] & & F \arrow[dd] \\
X \otimes A \arrow[ur] \arrow[d] & & & \\
X \otimes B \arrow[uurrr, bend left] \arrow[dr] & & & \\
 & Y \otimes B \arrow[rr] \arrow[uurr, dashed] & & G
\end{tikzcd}$$

Since $X \in (\mathcal{A}_f) = (\mathcal{A}_f)_{cof}$, $X \otimes A \rightarrowtail X \otimes B$, and these diagrams correspond bijectively to



$$\begin{tikzcd}
X \otimes B \sqcup_{X \otimes A} Y \otimes A \arrow[r] \arrow[d] & F \arrow[d] \\
Y \otimes B \arrow[r] \arrow[ur, dashed] & G
\end{tikzcd}$$

By LF given $X \otimes B \sqcup_{X \otimes A} Y \otimes A \overset{\sim}{\rightarrowtail} Y \otimes B$ is a equicofibration since $A \overset{\sim}{\rightarrowtail} B$ is.

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But then by definition of fibration a dotted arrow exists in the bottom square.

~~B~~ Under the correspondence this gives the dotted arrow exists in the upper square, as required to show the lifting property. QED.

Lets see ~~the~~ ~~thus~~ if the new model category fixes the complaint of Quillen in Homotopical Algebra II 6.3, II 4.1, 4.2

Let $\underline{A}$ be a category with finite limits and finite colimits (what Quillen means by closed under finite limits)

Suppose ~~A~~ $\forall X \in \underline{A}$ $\exists$ a projective object $P \in \underline{A}$ and an effective epimorphism $P \to X$ (Quillen: $\underline{A}$ has enough projective)

Let $s\underline{A}$ be the category of simplicial objects over $\underline{A}$

It is enriched over $\underline{\Delta}^{or}$-sets with cotensor and tensor over $/_f$

by ~~Quillen~~ Homotopical Algebra, II 1.7-1.8

Say a morphism $A. \xrightarrow{f} B.$ is

~~It is a cofibration if~~

1) a fibration if $\forall P \in \underline{A}$ projective

$$\underline{Hom}(P, A.) \longrightarrow \underline{Hom}(P, B.) \text{ is a Kan fibration}$$

2) a equimorphism if $\forall P \in \underline{A}$ projective

$$\underline{Hom}(P, A) \longrightarrow \underline{Hom}(P, B) \text{ is a weak equivalence in } \Delta^{\circ}\text{-sets.}$$

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NO

3) a cofibration of $A_\cdot \longrightarrow B_\cdot$ is a monomorphism with ~~B.~~ B_n/A_n a projective object.

Check this gives a ~~the~~ model structure satisfying the enriched ~~hypothe~~ hypotheses.

Check Basic model axioms 78 p156-157

B1 clear

B2 clear.

B3 clear

B4. Given

$$\begin{tikzcd} A_\cdot \arrow[r, tail] \arrow[d] & B_\cdot \\ C_\cdot & \end{tikzcd}$$

the pushout $C_\cdot \sqcup_{A_\cdot} B_\cdot$ exists as $\underline{A}$, hence $s\underline{A}$ has all finite ~~[illegible]~~ colimits.

The coker $C_\cdot \sqcup_{A_\cdot} B_\cdot / C_\cdot \cong B_\cdot / A_\cdot$ and so is projective in each degree.

To conclude $C_\cdot \longrightarrow C_\cdot \sqcup_{A_\cdot} B_\cdot$ is a cofibration, it remains to check it is mono.

Arsh! No reason why it should be mono.

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OK Choose definition of cofibration.

Go back & Quillen's

$A \longrightarrow B$ is a cofibration iff it has the left lifting property for the equifibrations.

Restart.

B1, B2, B3 still ok

B4: ~~Clear from definition~~

half LB3: sA has all finite colimits as A does.

So PO's exist.

Clear from definition that PO of a cofibration is a cofibration. Lifting property is preserved.

Sum

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & D & \downarrow \\ C & \longrightarrow & D \end{array}$$

as $A \xrightarrow{\sim} B$.

Why is $C \xrightarrow{\sim} D$?

- $\mathrm{Hom}(P, C) \xrightarrow{\sim} \mathrm{Hom}(P, D)$.

No good reason. LB3 shd

RB4 ok. Pullbacks exist, as $\mathrm{Hom}(P, \ )$ converts pullbacks to pullbacks. As $\Delta^\circ \mathrm{Set}$ is a proper model cat, the sim. RB4.

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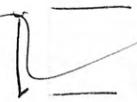
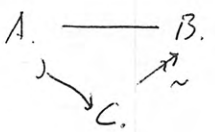
B5 trivial

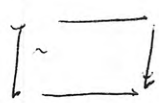
B6  has fill in by def.

B7 Quillen II 4.5 Pr 3



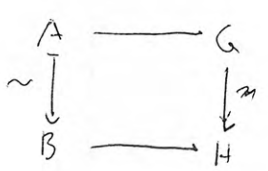
This gives half of B7. Put off other half

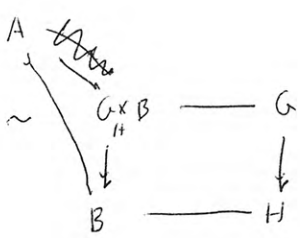
B6 has admit fill in by def.

Given 

Would like to use trick of 85 v165, dualize

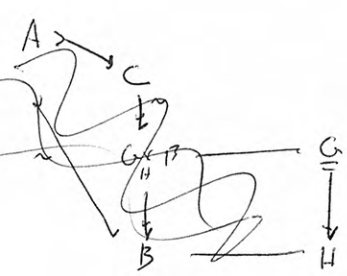
Pull by

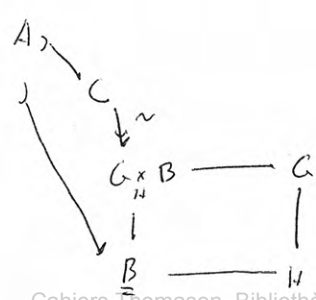




have on RB4

half of B7 we have so





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no need the other factors ($\xrightarrow{\sim}$ $\twoheadrightarrow$).

No go back to first idea

$A_\cdot \longrightarrow B_\cdot$ a cofibration iff

$\forall n$ $A_n \longrightarrow B_n$ is mono with B_n/A_n projective

Better $A_\cdot \longrightarrow B_\cdot$ a cofib iff

$\forall n$ $B_n \cong A_n \underset{\emptyset}{\cup} P_n$ with P_n projective

(equivalent in case A additive, but not in general.

Now easy to check ~~[illegible]~~ ~~[illegible]~~ hose chain preserves cofibration

$$\begin{array}{ccc} A_\cdot & \longrightarrow & B_\cdot \\ | & & | \\ C_\cdot & \longrightarrow & C \underset{A}{\cup} B \end{array}$$

$C_\cdot \longrightarrow C \underset{A}{\cup} B_\cdot$ is $C_n \longrightarrow C_n \underset{\emptyset}{\cup} P_n$!!

~~If $A_\cdot \xrightarrow{\sim} B_\cdot$ ...~~

But properties still a ~~[illegible]~~ a problem

Nec. $A_n \longrightarrow A_n \cup P_n = B_\cdot$ . . . $A_\cdot \xrightarrow{\sim} B_\cdot$

th $P_n = B_\cdot / A_\cdot$ is $\simeq 0$.

$$0 \longrightarrow \mathrm{Hom}(Q, A_n) \longrightarrow \mathrm{Hom}(Q, A_n \underset{\emptyset}{\cup} P_n) \longrightarrow \mathrm{Hom}(Q, P_n) \longrightarrow 0$$

NOT "EXACT" in non-additive case

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Somewhat VAGUE
IDEA

Better solution to Quillen's complaint argum

$s\underline{A}$: ~~Let~~

Consider $\underline{A}$ as a site with the canonical topology

Note P projectives give ~~a~~ with trivial topology

give equivalent topos.

Now $s\underline{A} \in \Delta^{op} A^{\wedge} \hookrightarrow \Delta^{op} P^{\wedge} \simeq \Delta^{op}$ (sheaves on A)

which is a topos by Joyal.

Quillen structure is restriction of structure of Joyal.

Only Problem is to try to show $s\underline{A}$ is

closed under the factorization which exists Δ^{op} (sheaves on $\underline{A}$)

Other problems

$\underline{\underline{C}}$ a site $\mathcal{C}$ model category satisfies rich hyp

~~Cat(C~~ $\# \underline{\underline{C}} < \aleph_1$,

Suppose $\mathcal{C}$ as $\aleph$-products + ω-summ

$\mathcal{C}$ as $\aleph$-colimits + $\aleph$-limits!

Then $Cat(\underline{\underline{C}}^{op}, \mathcal{C})$ has model structure

satisfies rich ~~pr~~ hypothesis.

$$Sheave(\underline{\underline{C}}, \mathcal{C}) \subseteq Cat(\underline{\underline{C}}^{op}, \mathcal{C})$$

F is $\{U_i - U\}$ a cover

$$FU - \prod FU_i \rightrightarrows \prod F(U_i \times_U U_j)$$

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Can we make a sheafification functor, and lift model ~~structure~~ structure to Sheaves.

~~More~~ ~~so~~

Surr. $\mathcal{A} \underset{i}{\overset{sh}{\rightleftarrows}} \mathcal{C}$

~~i~~ $sh \dashv i$ $\quad sh \circ i \xrightarrow{\cong} Id$ iso. $\Rightarrow$ i fully faith

$\mathcal{C}$ has a ~~enriched~~ model structure satisfying the ~~rich~~ rich hypothesis.

Can we ~~fit~~ lift to $\mathcal{C}$?

~~A~~ $\underline{Hom}_{\mathcal{A}}(A, A') = \underline{Hom}_{\mathcal{C}}(iA, iA')$?

$\cancel{K \otimes_{\mathcal{A}} A = sh(K \otimes_{\mathcal{C}} iA)}$

$K \otimes_{\mathcal{A}} A = sh(K \otimes_{\mathcal{C}} iA)$

$K \otimes_{\mathcal{A}} A = sh(K \otimes_{\mathcal{C}} A)$

Th $\cancel{\underline{Hom}_{\mathcal{A}}(sh(K \otimes_{\mathcal{C}} iA), A')} = \underline{Hom}_{\mathcal{C}}(i \circ sh(K \otimes_{\mathcal{C}} iA), iA')$

$\underline{Hom}_{\mathcal{A}}(sh\, C, A') = \underline{Hom}_{\mathcal{C}}(i \circ sh\, C, iA')$

~~ms~~ $\underline{Hom}_{\mathcal{C}}(C, iA')$?

No. st $K \otimes_{\mathcal{A}} A = sh(K \otimes_{\mathcal{C}} iA)$.

The ~~Hom~~ defn $\underline{Hom}_{\mathcal{A}}$ is $\underline{Hom}_{\mathcal{A}}(A, A')_n = Mor_{\mathcal{A}}(\Delta[n] \otimes A, A')$

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$$= \underline{\mathrm{Mor}}_e\left(i(\Delta[n] \otimes_a A),\ iA' \right)$$

$$= \mathrm{Mor}_e\left(i \cdot sh(\Delta[n] \otimes_e iA),\ iA' \right)$$

$$= \mathrm{Mor}_e\left(\Delta[n] \otimes_e iA),\ iA' \right)$$

$$\underline{\mathrm{Hom}}_a(A, A')_n = \mathrm{Mor}_a(\Delta[n] \otimes_e A,\ A')$$

$$= \mathrm{Mor}_a(sh(\Delta[n] \otimes_e iA),\ A')$$

$$= \mathrm{Mor}_e(\Delta[n] \otimes iA,\ iA') \qquad sh \dashv i$$

$$= \underline{\mathrm{Hom}}_e(iA,\ iA')_n.$$

So a consequence of defining $K \otimes_a A = sh(K \otimes_e iA)$

$$\therefore \underline{\mathrm{Hom}}_a(A, A') = \underline{\mathrm{Hom}}_e(iA, iA').$$

Now $\cancel{\underline{\mathrm{Hom}}_e(iA,}\ i$

$$\underline{\mathrm{Mor}}(K, \underline{\mathrm{Hom}}_a(A, A')) = \underline{\mathrm{Mor}}(K, \underline{\mathrm{Hom}}_e(iA, iA')$$

$$= \underline{\mathrm{Hom}}_e(iA,\ K \otimes iA')$$

Need right ~~exact~~ adjoint functor.

B. i: sheaves — ~~preserve~~ presheaves

~~has~~ But sh: not a right ~~exact~~ adjoint of anything: does not preserve infinite products.

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